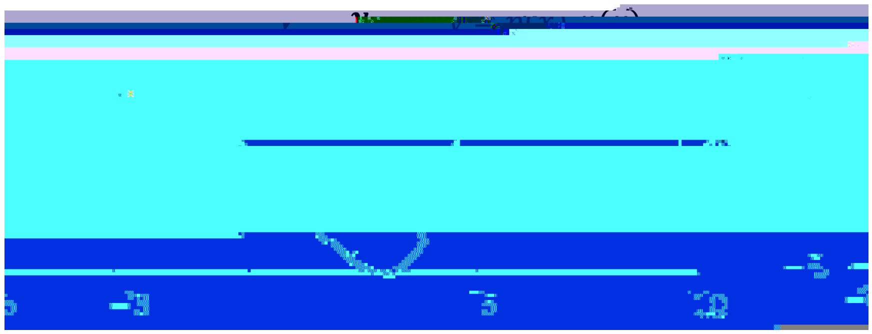
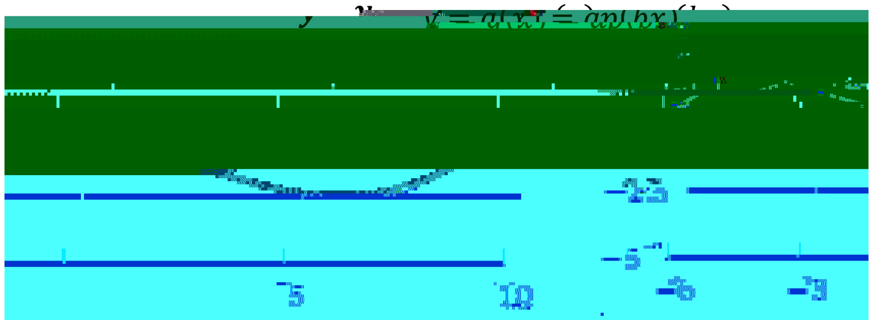
1. (20 pts) Parts (a) and (b) are not related.

   (a) For $f(x) = \frac{1}{x - 1}$ and $g(x) = \sqrt{2 - x}$

(b) The graphs below depict the functions $y = p(x)$ and $y = q(x)$, where q is a transformation of p of the form $q(x) = ap(bx)$. Find the values of a and b.





Solution:

The vertical difference between the maximum and minimum values of the curve for $p(x)$ is $3 - (-5) = 8$, while the vertical difference between the maximum and minimum values of the curve for $q(x)$ is $1.5 - (-2.5) = 4$. Therefore, the curve for $q(x)$ has been constructed by applying a vertical contraction of a factor of 2 to the curve for $p(x)$. This implies that $\boxed{a = 1/2}$

The horizontal difference between the endpoints of the curve for $p(x)$ is $5 - (-3) = 8$, while the horizontal difference between the endpoints of the curve for $q(x)$ is $10 - (-6) = 16$. Therefore, the curve for $q(x)$ has been constructed by applying a horizontal expansion of a factor of 2 to the curve for $p(x)$. This implies that $\boxed{b = 1/2}$

Note that $q(x) = 0.5\,p(0.5x)$.

(b) $\lim_{x \to 2} \frac{\sqrt{x+1} - \sqrt{3}}{x^2 + x - 6}$

Solution:

Begin by multiplying the numerator and the denominator by the conjugate of the original numerator expression.

$$\lim_{x \to 2} \frac{\sqrt{x+1} - \sqrt{3}}{x^2 + x - 6} = \lim_{x \to 2} \frac{\sqrt{x+1} - \sqrt{3}}{x^2 + x - 6} \cdot \frac{\sqrt{x+1} + \sqrt{3}}{\sqrt{x+1} + \sqrt{3}}$$

$$= \lim_{x \to 2} \frac{(\sqrt{x+1})^2 - (\sqrt{3})^2}{(x-2)(x+3)(\sqrt{x+1} + \sqrt{3})}$$

$$= \lim_{x \to 2} \frac{(x+1) - 3}{(x-2)(x+3)(\sqrt{x+1} + \sqrt{3})}$$

$$= \lim_{x \to 2} \frac{(x-2)}{(x-2)(x+3)(\sqrt{x+1} + \sqrt{3})}$$

$$= \lim_{x \to 2} \frac{1}{(x+3)(\sqrt{x+1} + \sqrt{3})}$$

$$= \frac{1}{(2+3)(\sqrt{2+1} + \sqrt{3})} = \frac{1}{(5)(2\sqrt{3})} = \boxed{\frac{1}{10\sqrt{3}}}$$

(c) $\lim_{x \to 0} x^4 \cos\left(\frac{1}{2x}\right)$

Solution:

$$-1 \le \cos\left(\frac{1}{2x}\right) \le 1$$

$$-x^4 \le x^4 \cos\left(\frac{1}{2x}\right) \le x^4$$ (Since x^4 is nonnegative for all x, the inequalities do not reverse direction)

$$\lim_{x \to 0}(-x^4) = \lim_{x \to 0} x^4 = 0$$

Therefore, the Squeeze Theorem indicates that $\lim_{x \to 0} x^4 \cos\left(\frac{1}{2x}\right) = \boxed{0}$

3. (30 pts) Consider the rational function $r(x) = \quad 3x^2 + 21$

(c) Find the equation of each vertical asymptote of $y = r(x)$, if any exist. Support your answer in terms of your work in part (b).

Solution:

The finite value of $\lim_{x \to -5} r(x) = -\frac{9}{8}$ determined in part (b) indicates that there is no vertical asymptote at $x = -5$.

The infinite limits $\lim_{x \to 3^-} r(x) = \infty$ and $\lim_{x \to 3^+} r(x) = -\infty$ were determined in part (b). Either one of those limits being infinite is sufficient to conclude that the line $\boxed{x = 3}$ is a vertical asymptote of the curve $y = r(x)$.

(d) Find the equation of each horizontal asymptote of $y = r(x)$, if any exist. Support your answer by evaluating the appropriate limits. *(Reminder: You may not use L'Hôpital's Rule or dominance of powers arguments to evaluate limits on this exam.)*

Solution:

$$\lim_{x \to -\infty} r(x) = \lim_{x \to -\infty} -\frac{3x^2 + 21x + 30}{x^2 + 2x - 15} = \lim_{x \to -\infty} -\frac{3x^2 + 21x + 30}{x^2 + 2x - 15} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x \to -\infty} -\frac{3 + 21/x + 30/x^2}{1 + 2/x - 15/x^2} = -\frac{3 + 0 + 0}{1 + 0 - 0} = -3$$

Therefore, the equation of the only horizontal asymptote is $\boxed{y = -3}$

4. (20 pts) Parts (a) and (b) are not related.

(a) For what value of a is the following function $u(x)$ continuous at $x = 4$? Support your answer using the definition of continuity, which includes evaluating the appropriate limits.

$$u(x) = \begin{cases} \dfrac{x-4}{x^2-16} & ; \quad x < 4 \\[2ex] \dfrac{1}{a-x} & ; \quad x \geq 4 \end{cases}$$

Solution:

By the definition of continuity, $u(x)$ is continuous at $x = 4$ if $\lim\limits_{x \to 4^-} u(x) = \lim\limits_{x \to 4^+} u(x) = u(4)$.

$$\lim_{x \to 4^-} u(x) = \lim_{x \to 4^-} \frac{x-4}{x^2-16} = \lim_{x \to 4^-} \frac{x-4}{(x-4)(x+4)} = \lim_{x \to 4^-} \frac{1}{x+4} = \frac{1}{4+4} = \frac{1}{8}$$

$$\lim_{x \to 4^+} u(x) = \lim_{x \to 4^+} \frac{1}{a-x} = \frac{1}{a-4}$$

$$u(4) = \frac{1}{a-4}$$

Therefore, $u(x)$ is continuous at $x = 4$ if $\frac{1}{8} = \frac{1}{a-4}$, which occurs when $\boxed{a = 12}$

(b)