1. (24 pts) Consider

$$f(x) = \begin{cases} 2x + 1 & 0 \leq x \leq 1 \\ 4 - x^2 & 1 < x \leq 2 \end{cases}.$$

   (a) Sketch a graph of $y = f(x)$. (The axes should be clearly labeled.)

   (b) Is f differentiable at $x = 1$? (Justify your answer with the definition of the derivative.)

   (c) Does the Mean Value Theorem apply to $f(x)$ on [0

2. (22 pts) The following problems are not related.

   (a) The planet N'Var has acceleration due to gravity of -12 meters per second squared. Suppose a stone is thrown from the top of a 42 meter tall building on the planet N'Var with an initial upward velocity of 36 meters per second. What is the velocity of the stone when it strikes the ground?

   (b) Determine the absolute minimum and the absolute maximum of $f(x) = e^{x^3+3x^2}$ over $[-5, -1]$.

Solution:

   (a) We are given that $a(t) = -12$, $v(0) = 36$, and $s(0) = 42$. Using antidifferentiation and the given initial conditions, we find that $v(t) = -12t + 36$ and $s(t) = -6t^2 + 36t + 42$.
   We now need to find the time when the stone strikes the ground. So, we have

$$0 = s(t) = -6(t-7)(t+1),$$

   which yields a solution of $t = 7$. (We can disregard the negative solution, as this would be before the stone is thrown. That is, it is outside the domain of our functions.)
   So, the velocity of the stone when it strikes the ground is $v(7) = -48$ meters per second.

   (b) We see that $f'(x) = 3x(x+2)e^{x^3+3x^2}$, which exists for all x in $[-5, -1]$. We see that $f'(x) = 0$ when $x = -2, 0$, but $x = 0$ is not in our domain. So, the only critical number is $x = -2$. Checking the critical number and the endpoints, we see

$$\begin{aligned} f(-5) &= e^{-50} \\ f(-2) &= e^{4} \\ f(-1) &= e^{2}. \end{aligned}$$

   So, the absolute maximum is e^4 and the absolute minimum is e^{-50}.

3. (20 pts) Let $y(x) = \left(1 + \frac{1}{x^2}\right)^x$

   (a) Find the value of $y'(1)$.

   (b) Evaluate $\lim_{x\to\infty} y(x)$.

Solution:

   (a) Use logarithmic differentiation.

$$y(x) = \left(1 + \frac{1}{x^2}\right)^x$$

$$\ln y = \ln\left(1 + \frac{1}{x^2}\right)^x = x\ln\left(1 + \frac{1}{x^2}\right)$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[x\ln\left(1 + \frac{1}{x^2}\right)\right]$$

$$\frac{y'}{y} = x \cdot \frac{-\frac{2}{x^3}}{1+\frac{1}{x^2}} + \ln\left(1 + \frac{1}{x^2}\right) = -\frac{\frac{2}{x^2}}{1+\frac{1}{x^2}} + \ln\left(1 + \frac{1}{x^2}\right)$$

$$y' = y \left[\ln\left(1 + \frac{1}{x^2}\right) - \frac{2}{x^2 + 1} \right]$$

The term y in the preceding equation is evaluated at $x = 1$ as $y(1) = \left(1 + \frac{1}{1^2}\right)^1 = 2$. Therefore,

$$y'(1) = 2\left[\ln\left(1 + \frac{1}{1^2}\right) - \frac{2}{1^2 + 1}\right] = \boxed{2\,(\ln 2 - 1)}$$

Note that an alternative approach would be to recognize that

$$\left(1 + \frac{1}{x^2}\right)^x = e^{\ln\left[\left(1+\frac{1}{x^2}\right)^x\right]} = e^{x \ln\left(1+\frac{1}{x^2}\right)}$$

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6. (16 pts) A rectangle is growing, but its length is always twice its width. Initially, the width of the rectangle is 3 centimeters. Two minutes later the width is 5 centimeters. If the rate of change of the width is proportional to the width (that is, $\frac{dw}{dt} = kw$), find the rate of change of the area of the rectangle after ten minutes.

Solution:

Let $w(t)$ be the width of the rectangle (in centimeters) after t minutes. We know that $w(t) = w_0 e^{kt}$ where $w_0 = w(0) = 3$ and $w(2) = 5$. We can use the latter point to find that $k = \ln \sqrt{\frac{5}{3}}$. Thus, $w(t) = 3e^{\ln \sqrt{\frac{5}{3}}\, t}$. Further, the area of the rectangle is given by

$$A(t) = (2w(t))(w(t)) = 18e^{\ln\left(\frac{5}{3}\right)t}.$$

So, we have

$$A'(10) = 18 \cdot \ln\left(\frac{5}{3}\right) e^{\ln\left(\frac{5}{3}\right)10} \text{ centimeters squared per minute.}$$

7. (16 pts) Sketch a graph of a single function $y = f(x)$ with all of the following properties:

- The domain of f is $(-3, 3)$.
- $\lim_{x \to a} f(x) = f(a)$ for all a except $a = 0$
- $f(1) = 0$
- $f(-2) = 0$
- $\lim_{x \to 3} f(x) = \infty$
- $\lim_{x \to 0} f(x) = -\infty$
- $f''(x) > 0$ for x in $(0, 3)$
- f is an odd function

Solution: