

Answer the following problems and simplify your answers.

1. (18pts) Find the **explicit solution** to the following initial value problem:

$$\begin{aligned} & \frac{dz}{dt} e^{t+z} = 0 \\ & z(0) = \ln 2 \end{aligned}$$

**Solution:** Using separation of variables, we have

$$e^z dz = -e^{-t} dt \Rightarrow e^z = e^{-t} + C :$$

Solving for  $z$  yields

$$z = \ln(C - e^{-t}); \quad C = C :$$

Applying initial conditions, we have

$$\ln 2 = \ln(C - 1) \Rightarrow C = \frac{3}{2} :$$

Then, putting everything together, we have

$$z = \ln \left( \frac{3}{2} - e^{-t} \right) :$$

2. (18 pts) Consider the curve  $y = \frac{x^3}{6} + \frac{1}{2x}$  on the interval  $\frac{1}{2} \leq x \leq 1$ .

- (a) Find the area of the surface obtained by rotating the curve about the  $y$ -axis.  
 (b) Set up, **but do not evaluate**, the integral with respect to  $x$  to find the area of the surface rotated about  $y = 2$ .

**Solution:**

- (a) First, we compute

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

Next, we can compute our length element as

$$\begin{aligned} ds &= \sqrt{1 + (y')^2} dx = \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx \\ &= \sqrt{1 + \frac{x^4}{2} - \frac{1}{2} + \frac{1}{2x^4}} dx \\ &= \sqrt{\frac{x^4}{2} + \frac{1}{2} + \frac{1}{2x^4}} dx \\ &= \sqrt{\frac{x^4 + 1 + \frac{1}{x^4}}{2}} dx \\ &= \frac{x^2}{\sqrt{2}} + \frac{1}{\sqrt{2}x^2} dx \end{aligned}$$

Since we are rotating the curve

3. (40 pts) Consider the region  $R$  bounded by  $y = \frac{1}{2}x^2$  and  $y = \sqrt{2x}$ .
- (a) Sketch and shade  $R$

respectively. Plugging these values in the washer method formula gives our volume as

$$V = \int_0^2 (R^2 - r^2) dx = \int_0^2 \left( 2 - \frac{x^2}{2} \right)^2 dx:$$

- iii. The base length of each rectangle is given by the vertical distance in  $R$ . In this case, the base

$$b = \frac{2 - x^2}{2}:$$

Then, the height of the region is  $h = 3b$  meaning the area of each rectangle is

$$A(x) = b \cdot h = 3 \cdot \frac{2 - x^2}{2}:$$

4. (24 pts) Determine whether or not the following sequences converge or diverge. Justify your answer! If the sequence converges, find its limit.

(a)  $\frac{(-1)^{n+1}n}{n^3-2+n}$       (b)  $\ln(2n^2+1) - 2\ln(n+1)$       (c)  $1 + 4^n - 3^{2-n}$

**Solution:**

- (a) First, we compute

$$\lim_{n \rightarrow \infty} \frac{(-1)^{n+1}n}{n^3-2+n} = \lim_{n \rightarrow \infty} \frac{n}{n^3-2+n} = \lim_{n \rightarrow \infty} \frac{1}{1+1/n} = \frac{0}{1+0} = 0$$

Since the absolute value of the sequence converges to zero,

$$\boxed{\lim_{n \rightarrow \infty} \frac{(-1)^{n+1}n}{n^3-2+n} = 0}$$

Finally, since the limit exists and is finite, the sequence **converges**:

- (b) Using log rules and continuity, we can compute our limit as

$$\lim_{n \rightarrow \infty} (\ln(2n^2+1) - 2\ln(n+1)) = \lim_{n \rightarrow \infty} \ln \frac{2n^2+1}{(n+1)^2} = \ln \lim_{n \rightarrow \infty} \frac{2+1/n^2}{(1+1/n)^2} = \boxed{\ln 2}$$

Since the limit exists and is finite, the sequence **converges**.

- (c) A little algebra yields

$$1 + 4^n - 3^{2-n} = 1 + 3^2 \frac{4^n}{3^n} = 1 + 9 \left(\frac{4}{3}\right)^n$$

The last term in our sequence is geometric with  $r = 4/3$ . Since  $4/3 > 1$ ,

$$\left(\frac{4}{3}\right)^n \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

meaning the original sequence **diverges** to infinity.