Fast algorithms for Helmholtz Green's functions

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2 , , 030 -0 2 ,

T f r a r r a i f a i- r i H Gr ’ f i l a i l

B i E a ' (1921) , r a r r a ii r a f r i n r i g
a va a i g (1.6) (Ga r & Z r (1980)a E (1998)f r a i rv

Proposition 2.1. ( ) $\in \mathcal{S}(\mathbb{R})$, Λ

, Λ^* .

3. Quasi-periodic Green's function via absolutely convergent series

T a i- i i Gr ' f i f r a il l (1.6) r i r a r , a i r v vi g i f i fr va ri a , g.

( ), a v i (1.1) gi v a a i a i f (1.4) a (1.5).

W i r v a

Proposition 3.1. *(3.1)* , *(1.2)* *(1.3)* > 0, $\neq 2$ $\boldsymbol{d}$K , $\boldsymbol{d} \in \Lambda^*$ $\in \mathbb{R}$. ≥ 2.

. T a i i ri i i i f r F ri r i (3.1) f f r

$$\text{F ri r C} = \frac{1}{} \sum_{\boldsymbol{d} \in \Lambda^*} \frac{\left(\frac{K\,2\ \boldsymbol{d}K\,{}^2 C\ {}^2}{4\ {}^2} \right)}{2\ \boldsymbol{d}K\ {}^2 K\ {}^2} \text{ i } \$ 2\ \boldsymbol{d}K \quad Ki\ \$ = Ki\ \$ \text{ F ri r} ,$$

3 3 1

i ${}^{2\,i\,\$\boldsymbol{d}} = 1$ f r a $\in \Lambda$ a $\boldsymbol{d}$

r

$$= \frac{1}{2}$$

Remark 3.3. W a iv ,i i f i i r r a a r f fr - a Gr ' f i fr

I r r i (3.12), a g r r f a i a i g r a i
i > . W P i a i f r a i r i i 2.1
l a i

$$\frac{1}{2^{3/2}} \quad 0 \quad \frac{Ci \quad ^{2}}{4 \quad ^{2}} \quad i \ \$ \quad K \quad C \ ^{2} \ ^{2}$$
$$\in \Lambda$$

$$= \frac{1}{2} \quad 0 \quad \frac{Ci \quad ^{2}}{4 \quad ^{2}} \quad \sum_{\boldsymbol{d} \in \Lambda^*} i \ \$ \ 2 \ \boldsymbol{d} K \quad K \frac{2 \quad \boldsymbol{d} K \quad ^{2}}{4 \quad ^{2}} \quad \frac{}{3},$$

r Λ^* i r i r a a i . B a g a i i i g r r f a i a
i g r a i , a r r i v a

$$\frac{1}{2} \sum_{\boldsymbol{d} \in \Lambda^*} i \ \$ \ 2 \ \boldsymbol{d} K \quad 0 \quad \frac{K \ 2 \quad \boldsymbol{d} K \quad ^{2} \ C \quad Ci}{4 \quad ^{2}} \quad \frac{}{3}$$

$$= \frac{1}{} \sum_{\boldsymbol{d} \in \Lambda^*} \frac{i \ \$ \ 2 \ \boldsymbol{d} K \quad \frac{K \ 2 \ \boldsymbol{d} K \ ^{2}}{}}{}$$

E T 2 4 7 . 5 . 5 6 0 . 5 4 6 9 0 j/ F 2 1 T 7 . 6 7 1 1

4. Fast convolutions with Green's function

R r a i f a i ri i Gr ' f i a a f i (3.1) a (3.2) i a a a a g f r i a i a i a a i . W r a a i a r r a i a r i a i g i g i (3.2) a a f G a . U g r i g a r i a i f Gr ' f i , r a a i a (r r)f ri a i a i . W r a g a r, i a i a i a i . W i a g r i g a .

L i a i a a r i a i f a i i Gr ' f i (3.4).

O i g i a a f r i F r, r a F r

$$\tilde{}_{F\ r} = \frac{1}{} \sum_{\substack{d\in\Lambda^* \\ 2\ dK \leq}} \frac{\left(\frac{K\ 2\ dK\ ^2 C\ ^2}{4\ ^2}\right)\ i\ \$\ 2\ dK}{2\ dK\ ^2 K\ ^2}, \tag{4.1}$$

r a r > 0 a > 0 a i f i a r r i a i r a .

F r a i a r a i i a r r a i a g a i i g a f i r a , a i i , r a a r i a i f i g i (3.3) a a f G a . F r a a r a g i a , > 0 r a (3.2) a

$$\sum_{\substack{\in\Lambda \\ \leq}} i\ \$\ _{i\ g}\ C\ ,$$

a i f i a r r i a . T , f r , a r i a i g a i B i . (i) i g a i r i a i f i g . T , a i a a r i a i f i g a a f G a ,

$$_{i\ g} = \sum_{=1} K\ ^2, \tag{4.2}$$

r > 0 a > 0. T ig i i a a r (B i . (i) f r a i). U i g (4.2), a r i a a i a a

$$\tilde{}_{a\ i a} = \sum_{\substack{\in\Lambda \\ \leq}} i\ \$\ _{i\ g}\ C\ . \tag{4.3}$$

C l i i g (4.1) a (4.3), a i i i Gr ' f i i a r i a a

$$\tilde{} = \tilde{}_{a\ ia} \quad C \tilde{}_{F\ ir\ r} . \qquad 4.4$$

W a r a r r f rr ri i a r i a i : (i) a r a i rr r r a i g i i i l i a (ii) a a r i a i rr r i r l (4.2). O i g i a a f r i l i , l r f ig i a r g a i i a i r a r .

W v i i ~F i r i F i r a i a

$$\tilde{}_{F\ ir\ r^{*}} = \frac{1}{} \sum_{\substack{d \in \Lambda^{*} \\ 2\ dK \leq}} \frac{d\ dK \quad \times \quad 2\ 2 \quad d \quad K}{\left(\frac{K\ 2\ \ dK\ ^{2} C\ ^{2}}{4\ ^{2}}\right)}$$

W a > 0 a > 1 a

$$\frac{1}{} \quad \sum_{\substack{d \in \Lambda^* \\ 2 \ d > }} \frac{\left(\frac{2 \ d \ ^2 C \ ^2}{4 \ ^2}\right)}{2 \ d \ ^2 \ ^2} \leq \frac{}{3}$$

a , ,

$$\left\| F \quad \tilde{} F \right\|_{1} \leq \frac{}{3}. \qquad 4.8$$

W i a a i a rr r

| a i a ~ a i a | 1

Wi i i f , a i (3.1) i a r r , , ≥ a i f

$$\frac{\left(\frac{K \; {}^2K^2}{4^2}\right)}{{}^2K^2} \le \frac{}{{}^2 \; {}^2K1}.$$

Wi gi v l (4.16), a i a r a i a r r a ril i f i a r r i (3.2) i l i r a r . A g r i r > 1, i r i i f i a r r a . F r r i ~3, f r a g a a a r a f r a a g r.

Remark 4.2. Diff r i f a v l a i v r a r i i g E a ' a i (g. C i (1978) r J r a . (1986) f r = 0). W i i (a M r 2006; Or a r . 2006 r B i . l i) a i f a i i a a a i r i g i a f a r . F r a , i g

(iv) . T f i ,a f ia ia
a F ir r ril i ,i va a a r f i .
W i ia ia ril i a r a i a a a iv r r if
a g ir f r B i . (2008), F ir r ril i r
i $\mathcal{O}$( ) i r i r N i a i g r . T ,
r a r ia i r a , ~ . Aga i , i r a
a v ~ C 2 g K1 .

A i g r r:

(i) ~ ia ia . U i g a g ir i B i . (2008),
i f a i g ia ia i (4.7) i $\mathcal{O}$ \$ \$ \$ i . A r a iv ,
fa G r f r (Gr ga r & S r i 1991; S r i 1991;
Gr ga r & S 1998) a , i r i a i ia r K1 a 3 -
i a i . A g \$ i f r a i a a \$ ~ g ,
a i i r g f a r r ir i , i
r a v ff iv a a a (v r i a i i , i a r ,
fa a a g ir i B i . (2008) a
Ga ia r a a). N a i (4.7) r = 0 i a
a i a i i i r ril i g a
i a i r i r r a i f r r. W a v a ,
a i a i f i g (4.7) i $\mathcal{O}$ C 3 g K1),
r 3 i a a .

(ii) F ir r. W va a F ir r r f r f i
f i a r i r a a i i i i r 2 d K ≤
a i r a r. W a ~ g K1
ia a f r i (3.1). G v a f a i
va a (4.5), USFFT (D & R i 1993; B i 1995;
L & Gr ga r 2005) va a rig ir . T ,
a i a i i $\mathcal{O}$ C C $\mathcal{O}$ g , r $\mathcal{O}$ g C
4 g K1 , r 4 i a a .

W a i f r a f l , ia ia a F ir r, f

W a r r a i i (4.17) a va a Gr '
f i a r vi a a g ri f ri a i a i a a r r. B
r a , r a r a r a Gr ' f i a a r r a r a
a r i a i a i a f a a a r a g ri f ri a i a i . F r
n f a ri , i va a i f Gr ', f i i
(4.17) l i g f i (4.17) a a i , ri i g
= Δ C . W (M P r a . 2000, (17)) Δ a
(Li 1998, (2.49), (2.53) a (2.54)) .
I g r 1, i a rr rl (4.17) a r a r i a i ~ i
(4.4) r f r a r a ≈ 10^{-9}. W a i r a a r = 0
i r f a r i a i g a a ff i a i a i a
a r r (a r a ≈ 10^{-9}) a i r i r i i 4.1.
N v rif a r a f r a g ri l i ri g a i ri i
f i

$$= \frac{2\alpha}{} \ldots{}^{3} \ldots -\alpha \ldots C \ldots{}^{2} \qquad 4.18$$

$$\in \Lambda \quad = 1$$

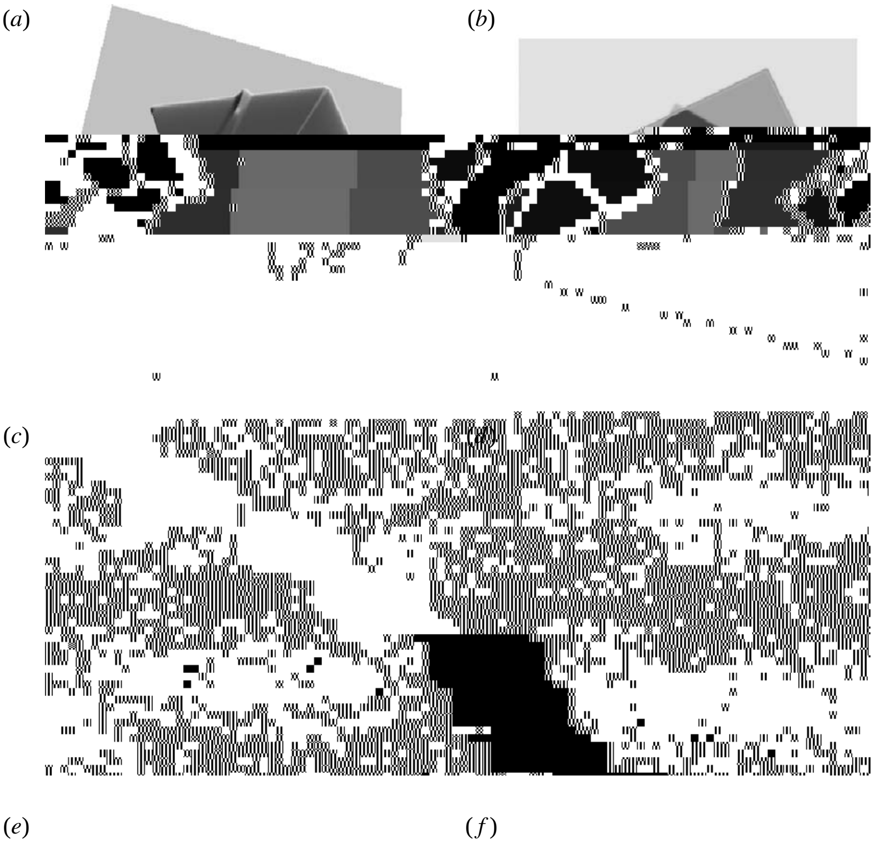
i a r r $\alpha = 300$, $= (1/3, 4/7)$, $_1 = (0, 0)$, $_2 = (1/10, 1/10)$ a $_3 = (-3/$

(a) (b)

Figure 3. A a i- i i Gr ' f i i = (3, 5) a | = 100 f r a - i i a a g a a i i a i v r $_1$ = 1, 0 a $_2$ = 1/2, 3/2 i r gi K1/2, 1/2 ! K1/2, 1/2 : ( ) a r a a r a ( ) a i a gi a r a r .

l a i a a F i r a r f a g i . I Figure 2, i a a rr r a g i a g a f i i v . Gr ' f i a a r i a i . = 10^{K11}, r a ´ ^{2}- r f i i $_2$ ≈ 1.76 a a f ig - a i i $_2$ ≈ 1.31$$10^3$. T i r a g r | | i i a i r i i 4.1. | |

N , f r f v vi g i v a i- i i Gr ' f i . I Figure 3, i a a i f a - i i a a i- i i Gr ' f i a a f i . T i v a i f r r i g i a i f : ( ) a r a r a i a i a l f i F i r f r a v a a ( ) i a Gr ' f i i f . I Figure 4, i a r f v vi g a i i Gr ' f i i a f a i r i a f i i j i i i . W a i a r i f (i i) f i .



5. Green's functions with boundary conditions on simple domains

W av ar r Gr ' f i a i r a
a r i i i a i i g r r f r a i-
i i Gr ' f i (3.4). W a a g r i g i g r,
r r a r g r v i , a g f r a i g Gr ',
f i i i i a r a f r a i- i Gr ' f i . T
a i a i f Gr ' f i a i f i g D , N a r i

l a r i i i a g a i i l a i a a F r r a i .
I a i a a i , a a r r r a i i v v i g G i a a
i F r r a i a a I l i a i f i i a i r r .
F r a f a i , i r - i i a a i D r
l a r i i r i i v = K1/2, 1/2 ! K1/2, 1/2 . W
r Gr ' f i i g r i Gr ' f i (i 2
i a f), a i f i g

$$\Delta C 4 \; ^2 \qquad = K$$

a (1.3) i = 0. W a f r a r i f r i Gr '
f i i i a i f f r

$$f \; r \; a \quad {}_1, \; {}_2 = K \frac{1}{4} \sum_{{}_1=K\infty}^{\infty} \sum_{{}_2=K\infty}^{\infty} \quad 0 \; 2 \quad {}_1 C \; {}_1 \; ^2 C \; {}_2 C \; {}_2 \; ^2 \quad ,$$

i , i (1.6), a i i a g i a r a r f f r - a
Gr ' f i i r , ${}_1$ = 1, 0 a ${}_2$ = 0, 1 .
W r v a f r i v r g r i (3.4),

$$\quad {}_1, \; {}_2 = \frac{1}{2} \sum_{\in \mathbb{Z}^2}^{\infty} \quad g \; 2 \quad K \quad C \quad ^2 \frac{2}{4} C \; 4 \; ^2 \; K2$$

$$C \sum_{\in \mathbb{Z}^2} \frac{K \; ^2 \quad ^2 C \; ^2}{\qquad}$$

r

, , = (K 4 K C 2 2) K (K 4 C C 1 C 2 2). 5.3

T , a i a i f r r (5.2) a r a g a i r i a

~D a i a , = =1 1/2 K1/2 , 2 2, 2 $^2_1$C $^2_2$≤

1/2 ! K1/2 , 1 1, 1 1, 2 1 2,

i a l a r f r r i g f a a g i r i l i §4.
I F i r r a i , f r a i r a r a , >1
a i f (4.8) a l a i

~D F i r 1, 2, 1, 2 = 2 $^2_1$C $^2_2$≤ K 2 $^2_1$C $^2_2$ C 2 2 / 4 2 $^2_1$ C $^2_2$ K 2 i 1 1 C 2 2

! (Ki 1 1 K i 1 1 C1) (Ki 2 2 K i 2 2 C1). 5.4

W a i r a r a

~D F i r , = 2 $^2_1$C $^2_2$≤ K 2 $^2_1$C $^2_2$ C 2 2 / 4 2 $^2_1$ C $^2_2$ K 2 i 1 1 C 2 2

! ^ 1, 2 K i 1 ^ K 1, 2

K i 2 ^ 1, K 2 C i 1 C 2 ^ K 1, K 2 ,

5.5

r ^ i g i v i (4.6). W USFFT v a a (5.5) a i §4 .

Remark 5.1. A r i l l

Remark 5.2. T r i f Gr ’ f i i Di r N a a r i i i i i = 3i a a g - i i a a a i f a i a i f ig r . I r a , i r a r i a i a v a f r i a i i . F r a ,i a i a a i a r i a i f Gr ’ f i i Di r a r i i i gi v

$$\tilde{D}$$

a i a 1, 2, 3, 1, 2, 3

$$= \frac{1}{2} \quad , 1 \quad 1, 1 \quad , 2 \quad 2, 2 \quad , 3 \quad 3, 3 ,$$

$${}^2_1 C \; {}^2_2 C \; {}^2_3 \leq \; = 1$$

r a r ri i (4.2) a , i (5.3). S i a r , a v a a a g f (5.4),

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