Iterated Spherical Means in Linearized Inverse Problems

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Abstract. We consider a representation of the function

$$F(x) = \frac{1}{(2\pi)^n} \int_{|p|<2k} \hat{F}(p) e^{ip\cdot x} dp,$$

in terms of the iterated spherical mean of $\hat{F}(p)$. Here, n is the dimension of the space. We also review applications of such a representation to linearized inverse problems and present

1. Introduction. Experiments in scattering usually yield the measured scattered field as a function of two unit vectors which represent the direction of propagation of the incident wave and the direction at which the field is recorded. In many cases (we provide

[illegible]

where $\omega_n = \dfrac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the surface area of the unit sphere in R^n, x is a point in R^n, ν is a unit vector in R^n and $d\omega_\nu$ is the standard measure on the unit sphere (the solid angle differential form), such that $\int_{|\nu|=1} d\omega_\nu = \omega_n$. The function $I(x,r)$ is the normalized average of the function f on a sphere of radius $|r|$ about the point x. We note that the function $I(x,r)$ is even with respect to r.

The iterated spherical mean $M(x, \alpha, \beta)$ is defined as follows

[illegible]

... are unit vectors in R^n. F. John [5] obtained the

[illegible]

If the support of $\hat{F}(p)$ is not restricted to the ball B_{2k} in (3.2), then (3.4) defines the low-pass-filtered version of the function whose Fourier transform is $\hat{F}(p)$. In this case (3.3) is the representation of the low-pass-filtered version of the function in terms of the iterated spherical mean of its Fourier transform.

The representation in (3.3) was derived in [4]. For n = 2 and n = 3 it reduces to formulae obtained by A. J. Devaney [1,3].

4. Inverse Scattering in Born Approximations. The most simple example of an application of the representation in (3.3) is the inversion formula for inverse scattering and

$$Q_{LP}(x) = \frac{k^3}{\pi^2} \int_{|\mu|=1} \int_{|\nu|=1} |\nu-\mu| \, |f(k,\nu,\mu)|^2 \, e^{i(k\nu-k\mu)\cdot x} d\omega_\nu \, d\omega_\mu . \qquad (4.6)$$

Thus, we obtain that if we can measure the phase of the scattering amplitude we have the explicit inversion formula in (4.5) for the reconstruction of the low-pass-filtered version of the potential. In the case when the phase of the scattering amplitude cannot be directly measured we can explicitly reconstruct the interatomic distance function using (4.6). Formulae (4.5) and (4.6) were first obtained in Refr. [3].

Remark 1: In the case of inverse scattering in the n-dimensional space one obtains the analogous result as soon as the scattering where

$$\Psi(p,k,\nu) = \int U(x,k)e^{-ip\cdot x}dx\,,$$

$\delta(p-k\nu)$ represents the incident plane wave, p is a vector in R^2, ν is a unit vector in R^2 and

$$\hat{O}(p) = \int O(x)e^{-ip\cdot x}dx\,.$$

[redacted]

initial plane wave. We set

$$p = \eta\nu + \xi\nu^{\perp}, \tag{5.3}$$

[redacted]

Let us consider the scattered field $\Psi_{sc}(p,k,\nu) = \Psi(p,k,\nu) - \delta(p-k\nu)$. We find

$$\Psi_{sc}(p,k,\nu) = \frac{k^2}{k^2 - |p|^2 - i0} \int \hat{O}(p-p')\,\Psi(p',k,\nu)dp'\,, \tag{5.4}$$

[redacted]

$$O_{LP}(x) = \frac{ik}{4\pi^3} \int\limits_{|\nu|=1} \int\limits_{|\mu|=1} (1-(\mu\cdot\nu)^2)^{1/2} |\mu\cdot\nu| \, e^{-ik|\mu\cdot\nu|y} \Psi^{b}_{sc}(y,\mu,k,\nu) e^{i(k\nu-k\mu)\cdot x} d\omega_\nu d\omega_\mu \, . \quad (5.9)$$

The formula (5.9) is a backpropagation inversion formula which was first obtained by A. J. Devaney [1] and is presented here in a slightly different form.

The case of Rytov approximation is analogous to Born approximation and can be found in Ref. [1]. We note that in the case of a plane incident