

Synchronization in large directed networks of coupled phase oscillators

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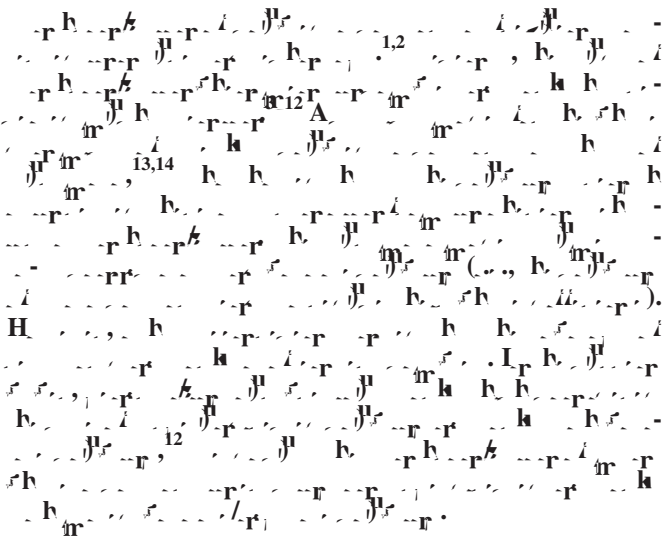
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Received 29 August 2005; accepted 11 November 2005; published online 31 March 2006

We study the emergence of collective synchronization in large directed networks of heterogeneous oscillators by generalizing the classical Kuramoto model of globally coupled phase oscillators to more realistic networks. We extend recent theoretical approximations describing the synchronization in large, undirected networks of coupled phase oscillators to the case of directed networks. We also consider the case of networks with mixed positive-negative coupling strength. We compare our theoretical numerical simulation and find good agreement. — 2005 American Institute of Physics. DOI: [10.1063/1.2148388](https://doi.org/10.1063/1.2148388)



I. INTRODUCTION

The classical Kuramoto model^{13,14} describes a collection of globally coupled phase oscillators that exhibit a transition from incoherence to synchronization as the coupling strength is increased past a critical value. Since real world networks typically have a more complex structure than all-to-all coupling,^{15,16} this natural network has effective interaction structure has an impact on the synchronization transition. In Ref. 12, we studied the Kuramoto model allowing general connectivity of the nodes, and found that for a large class of networks there is still a transition to global synchronization as the coupling strength exceeds a critical value k_c . We found that the critical coupling strength depends on the large eigenvalue of the

adjacency matrix A describing the network connectivity. We also developed a general approximation describing the behavior of an order parameter measuring the coherence of the transition. This paper addresses the case in which $A_{nm} = A_{mn}$. In a directed network in which the coupling ends do reduce the phase difference of the oscillators.

Most networks considered in applications are directed,^{15,16} which implies an asymmetric adjacency matrix, $A_{nm} \neq A_{mn}$. Also, in some cases the coupling between oscillators might drive them to be out of phase, which can be represented by allowing the coupling term between the oscillators to be negative, $A_{nm} < 0$. The effect of the presence of directed and mixed positive-negative connections can have on synchronization, therefore, of interest. Here we show how our previous theory can be generalized to account for the effect of the adjacency matrix or the effect of the negative connections are particularly interesting and compare our theoretical approximation with numerical simulation.

This paper is organized as follows. In Sec. II we review the results of Ref. 12 for undirected networks with positive

man the ergodicity of coupled phase oscillators. This equation can be modeled by the equation

$$\dot{\theta}_n = \omega_n + k \sum_{m=1}^N \sin(\theta_m - \theta_n)$$

A eraging o er he freq encie , one ob ain he *frequency distribution approximation* FDA

$$r_n = k \frac{A_{nm} r_m}{g z k r_m} \int_{-1}^1 \frac{1}{z^2} dz. \quad 13$$

The al e of he cri cal co pling reng h can be obtained from he freq enc di rib ion appro imation b le ing $r_n \rightarrow 0^+$, prod cing

$$r_n^0 = \frac{k}{k_0} A_{nm} r_m^0, \quad 14$$

here $k_0 = 2/g$. The cri cal co pling reng h h corre pond o

$$k_c = \frac{k_0}{2}, \quad 15$$

here i he large eigen al e of he adjacenc ma ri A and r^0 i propor ional o he corre ponding eigen ec or of A. B con idering per rba ion from he cri cal al e a $r_n = r_n^0 + r_n$, e panding $g z k r_m$ in Eq. 13 o econd order for mall arg men , m l ipl ing Eq. 13 b r_n^0 and , m ming o er n, e ob ained an e pre ion for he order param e er pa he ran ion alid for ne ork i h rela i el homogeneo degree di rib ion¹⁷

$$r^2 = \frac{1}{k_0^2} \frac{k}{k_c} \left(1 - \frac{k}{k_c} \right)^3, \quad 16$$

for $0 < k/k_c < 1 \ll 1$, he2r4115.078hj/F6.768598406.2813Tm2764444111..9.9789j/F599Tc-307.9h37859.di rib ion 6.913Tm276444

$$r = \sum_{n=1}^N r$$

hand, the TAT and the relative from direct numerical solution of Eq. 1 show dependence on the realization. Since the FDA and MFT incorporate the realization of the connection A_{nm} , both the frequencies, we interpret the observed realization dependence of the TAT and the direct solution of Eq. 1 as indicating that the latter dependence is due primarily to variation in the realization of the frequencies rather than to variation in the realization of A_{nm} .

We have for our example $N=1500$ and $s=2/15$ implying a average degree $d^{in} = d^{out} = 200$. The following comparison procedure, we generated an undirected network as follows: Starting with a Eq. a9F54825ek-2 in the TDform realization 4direct

the adjacent matrix is independent chosen to be 1 with probability s and 0 with probability $1-s$, and the diagonal elements are zero. Even though the network constructed in this way is directed, for most nodes $d_n^{in} \approx d_n^{out}$. For $N=1500$ and $s=2/15$, Fig. 1a shows the average of the order parameter r^2 obtained from numerical solution of Eq. 1 averaged over realization of the network and frequencies triangle, the frequency distribution approximation FDA, solid line, and the mean field theory MFT, long dashed line as a function of k/k_c , where the relative for the FDA and the MFT are averaged over the realization. The permutation theory Eq. 16 agreed with the frequency distribution approximation and a left over for clarity. The error bars correspond to one standard deviation of the sample of realization. We note that the larger error bars occur after the transition. When the values of the order parameter are averaged over realization of the network and the frequencies, the relative shows good agreement with the frequency distribution approximation and the directed mean field theory.

In order to describe the order parameter dependence in Fig. 1b the order parameter r^2 obtained from numerical solution of Eq. 1 for a particular realization of the network and frequencies both, the time averaged theory long dashed line, and the frequency distribution approximation solid line as a function of k/k_c . A can be observed from the figure, in contrast with the time averaged theory, the frequency distribution approximation deviates from the numerical solution both by a small but noticeable amount. This behavior is observed for the other realization as well. We note that the FDA and MFT relative are in fact identical for all realization. On the other

where, as in the undirected case, the value of the average of the order parameter obtained from numerical solution of Eq. 1. The directed percolation theory gives a good approximation for small values of k close to k_c , as expected. On the other hand, the directed mean field theory predicts a transition point which is smaller than the one actually observed.

When numerically solving Eq. 32 by iteration of Eq. 33, on some occasions a period of orbital bifurcation is found in each of the desired points. If we denote the left hand side of Eq. 33 by z_n^{j+1} and the right hand side by $f(z_n^j)$, we found convergence to a fixed point is facilitated by replacing the right hand side by $(z_n^j + f(z_n^j))/2$ and finding the fixed point of this modified system.

In this example, a low coupling strength $k/k_c \lesssim 4$, where k_c is computed from Eq. 37 the order parameter computed from numerical solution of Eq. 1 is smaller than has been obtained from the TAT and FDA. As k increases, however, the TAT and FDA theories capture the asymptotic value of the order parameter r . We note that in this case the asymptotic value is larger than has been corresponding to phase locking i.e., the one obtained by setting $\dot{\theta}_n = 0$ in Eq. 35, $r = 0.54 - 0.46 = 0.08$, which is indicated by a horizontal dashed line in Fig. 4, and much smaller than $r = 1$, the value corresponding to no frustration i.e., $\theta_n = m$ for $A_{nm} \rightarrow 0$ and for $A_{nm} = 0$ in Eq. 35. The small scale of the horizontal axis is due to the fact that we are plotting r^2 , and our definition of the order parameter which assigns a value of 1 to a nonfrustrated configuration. The small value of the order parameter indicates a strong frustration.

We note that in this example, in contrast to the example discussed so far, there is a variation in the value of the order parameter predicted by the FDA for different realizations of the network. This indicates that, as the expected value of the coupling strength A_{nm} becomes small i.e., $q = 1/2$ small, frustration of the realization of the network becomes noticeable. Although the value predicted by the FDA and TAT depend on the realization of the network and frequency, we note for $k/k_c \ll 6$ that the value of r^2 is observed for the numerical simulation of the corresponding realization. An illustration of this is plotted in Fig. 5 the value of r^2 obtained from the TAT and the value of r^2 obtained from the FDA diamond series, the value obtained from numerical solution of Eq. 1 for $k/k_c = 8$. Each point corresponds to a given realization of the network, with the larger error bars representing realizations of the frequency. The ellipse surrounding the TAT data has a horizontal and vertical half-width corresponding to the an-

standard deviation of r^2 TAT and r^2 simulation for the ensemble of realizations. The half-width of the horizontal bar on the diamond FDA data indicates the standard deviation of r^2 simulation.

lation in network with a much larger number of connections per node, the effect of correlation would likely be reduced.

We also considered a case in which the adjacency matrix is asymmetric and has mixed positive-negative connections. For $N=1500$ nodes, we considered an adjacency matrix being independent entries of 1, -1, and 0 with probabilities $8/45$, $4/45$, and $11/15$, respectively. The larger probability field an expected number of connections of 400. Our theoretical work is facilitated in this case, and, since the real parts are similar to those in Fig. 3, we do not show them. In this case there is no guarantee that there is a real eigenvalue needed for eliminating the critical coupling strength in Eq. 15, or that the large real eigenvalue if there is one has the large real part. Numerically, we find that for matrices considered in this example there is a real positive eigenvalue and that, furthermore, it is well separated from the large real part of the remaining eigenvalues (see Fig. 6). We also find that for other values of q provided $q > \frac{1}{2}$ is not too small. We provide a discussion of this issue and how the

of the non-ero enrie being chosen randoml e.g., in the symmetric case, the position of the non-ero enrie is chosen when constructing the network, using the congruion model, and their albe being al o d ermined randoml from a given probability distribution e.g., with probability q and with probability $1 - q$. Or in ere i focu ed on the gap between the large real eigen albe if here i one and the large real par of the o her eigen albe . In Ref. 23 the pec r m of certain large par e marice i h a erage eigen albe 0 and ro m $\sum_{m=1}^N A_{nm} = 1$ a de cribed and a heric anal ic al approach a propo ed. Using re l for marice i h ero mean Gaussian random enrie ,²⁴ Ref. 23 predic ha the pec r m of the non-Gaussian random marice he con ider con i of a ri al eigen albe = 1 i h the remaining eigen albe di rib ed , niforml in a circle centered a the origin of the comple plane i h radi

$$= \frac{1}{N} , \tag{A1}$$

here σ^2 i the ariance of the enrie of the marice . We nd ha hi approach al o , cceed in de cribing the pec r m of the marice in o r e ample . In o r ca e, the diagonal enrie are 0, o ha the a erage eigen albe i al o 0 a in Ref. 23. We nd ha here i al a large real eigen albe appro ima el gi en b the mean eld albe

$$= \frac{\tilde{d}^2}{\tilde{d}} \tag{A2}$$

ee Ref. 12 and 25 , here $\tilde{d}_n = \sum_{m=1}^N A_{nm}$ and $\tilde{d}^2 = \sum_{n=1}^N \tilde{d}_n^2$, hich in the ca e con ider ed in Ref. 23 red ce o = 1. We al o n mericall con rm ha the remaining eigen albe are , niforml di rib ed in a circle of radi a de cribed in Ref. 23. Thi i ill r a ed in Fig. 6.

Th for $N \gg 1$ if here i a gap of i e between the large real eigen albe and real par of the re of the eigen albe pec r m. Using Eq. A1 and A2 i can be hon ha , for neork i h large eno gh n mber of conec ion per node or i h eno gh po i e or nega i e bia in the copling reng h, here i a ide epara ion between the large eigen albe and the large real par of the remain-

ing eigen ec or . For symmetric marice , imilar re l appl i.e., the blk of the pec r m of the marice A can be appro ima el ob ained a de cribed abo e , using Wigner' emicircle la .

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