A Multiresolution Approach to Fast Summation and Regularization of Singular Operators

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B.Sc., Metropolitan State College, 1989
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A thesis submitted to the
Faculty of the Graduate School of

This thesis for the Doctor of Philosophy degree by
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has been approved for the
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Thesis directed by Professor Gregory Beylkin

A class of fast algorithms is introduced for the evaluation of discrete sums that utilizes projections on a multiresolution analysis. The discrete sums under consideration arise, for example, in the study of physical systems by means of particle simulations requiring long-range potentials. These include gravitational and electrostatic models, plasma physics, atmospheric physics, and vortex methods in fluid dynamics. These numerical models of particle interactions require the application of dense matrices hich, done directly, requires $O(N^2)$ arithmetic operations. The algorithms e develop accomplish this task to ithin accuracy ϵ in $O(N)$ arithmetic operations.

There are t o types of algorithms used today for the fast computation of discrete sums, namely, the Method of Local Corrections and the Fast Multipole Method. Our approach is related to both, but has its o n unique features. We describe implementations in one and t o dimensions, and present theoretical foundations for algorithms in higher dimensions.

In our approach to discrete summation problems, e construct explicit representations of singular operators on subspaces of the multiresolution analysis. These representations provide a definition for the regularization of such operators, as ell as a practical algorithm for their computation. We present a ne multiresolution approach to the regularization of singular operators, and sho that our method coincides ith the classical method, here the classical method is applicable.

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hapter 1

Introduction

1.1 Introductory Remarks

The study of physical systems by means of particle simulations is an important computational tool in many fields. Examples include plasma physics, atmospheric physics, N-body gravitational problems, and vortex methods in fluid dynamics. In most of these particle models the evaluation of discrete sums describing the pair ise interactions bet een particles occupies a central role, and is often the most expensive part of the computation. Discrete sums may also be encountered in the evaluation of integral equations obtained in the solution of boundary value problems. In any event, numerical models require the application of dense matrices hich, done directly, requires an amount of ork proportional to N^2 for an N particle system (or N point discretization). To overcome this computational hurdle there is a need for fast $O(N)$ algorithms.

In this thesis, e introduce a class of fast algorithms for the computation of discrete sums using projections on a multiresolution analysis. (A br ft **d** l l a

As a special case of (1.1) e consider the sums

$$g(x_m) = \sum_{\substack{n=1 \\ n \neq m}}^{N} K(x_m - x_n) f(x_n) \,, \qquad 1 \leq m \leq N \tag{1.2}$$

 here the N sampling locations coincide ith the positions of the N particles. Indeed, there is no loss of generality in taking (1.2) as our model problem, and this e do from no on. The particle locations $\{x_n\}_{n=1}^{N}$ are points in $\mathbf{R}^d$, here

In our approach to discrete summation problems, e construct explicit represeentat for some positive δ. It follo s from (1.7) that $|K_{HF}(x, y)|$ decays rapidly as $|x - y|$ increases. Therefore, this singular part of the kernel influences only the short range, or local interactions, and is represented by a banded matrix hich can be applied to a vector in $O(N)$ operations.

1.3 A B

Implicit in the FMM is the simple splitting $K = K_{LF} + K_{HF}$, here

$$K_{LF} = \begin{cases} 0\,, & |x-y| \le \delta \\ K(x-y)\,, & |x-y| > \delta \end{cases} \quad \text{and} \quad K_{HF} = \begin{cases} K(x-y)\,, & |x-y| \le \delta \\ 0\,, & |x-y| > \delta \end{cases}$$

The lo frequency part is applied by a clever use of multipole expansions in a divide and conquer strategy, taking advantage of the smoothness of the kernel on regions removed from the origin to truncate the expansions after a fe terms. The high frequency part is applied directly. Despite its apparent simplicity, the algorithm is not so straightfor ard to implement. The explicit form of the multipole expansions and translation operators, hich are responsible for shifting the expansions from one box center to another, must be orked out ane for each ne kernel. Ho ever, hen properly implemented, the FMM provides a very efficient $O(N)$ algorithm.

The Method of Local Corrections as introduced in [1] as a vortex method for solving problems in fluid mechanics, though the main ideas are certainly relevant in a more general context. This method is designed to approximate the velocity field due to a distribution of “vortex blobs” in a fluid, and to evaluate this field at the center of each blob. A vortex blob is a radially symmetric function, usually ith compact support, that approximates a point vortex. In [1] it is observed that “...the difference bet een the velocity field due to a point vortex and a vortex blob located at the same point in space becomes very small as one moves a ay from the center of the vortices.” (This statement should be compared to the error estimate (1.8).) The approximate velocity is first obtained on an equispaced grid via a fast Poisson solver (FFT), and then interpolated to the centers of the vortices. The approximation is then corrected locally, for all vortices that lie in close proximity to other vortices. The interpolation is accomplished by the use of a complex-valued interpolating polynomial, made possible by the fact that the $x-$ and $y-$ components of velocity are the real and imaginary parts of a harmonic function. This is a special feature of vortex methods, and is not likely to generalize to other applications. The correction step in this method is essentially equivalent to (1.7), and the splitting of the kernel is achieved by means analogous to that described in Section 1.2. Ho ever, in approximating the velocity field, a smoothed version of the original operator is not constructed, and a finite difference method is employed instead.

In [11], a class of algorithms for particle simulations involving Poisson's equation is developed. These algorithms are named PPPM by the authors, hich stands for “particle-particle, particle-mesh.” The name refers to the by no familiar splitting of the summation into lo and high frequency contributions. The lo frequency part is evaluated on an equispaced grid, or mesh. The values assigned to the mesh points are obtained from the charges on the particles by use of a “charge-sp

hapter 2

Multiresolution Analysis

2.1 Definition and Basic Properties

We first make some preliminary comments. Throughout this thesis, e use the notation $\hat{f}$ to refer to the Fourier transform of a function f,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{ix\cdot\xi} f(x)\, dx\,, \qquad f(x) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} e^{-ix\cdot\xi} \hat{f}(\xi)\, d\xi\,.$$

for a function $f \in L^2(\mathbb{R}^d)$. We use $(\cdot\,,\cdot)$ to refer to the usual inner product on $L^2(\mathbb{R}^d)$,

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g}(x)\, dx\,.$$

A superscript and a subscript on a function ill denote, respectively, a dilation and a translation,

$$f_k^j(x) = 2^{-dj/2} f(2^{-j}x - k)\,,$$

here $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. This ill someti

In hat follo s, e make repeated use of Poisson's summation formula,

$$\sum_{k=-\infty}^{\infty} f(k)e^{ik\xi} = \sum_{l=-\infty}^{\infty} \hat{f}(\xi + 2l\pi)\,. \tag{2.1}$$

There exist many proofs of this ell-kno n result (see e.g. [18] or [22]).

The follo ing definition is by no standard, and is borro ed from [20, p.21].

Definition 2.1.1 *A multiresolution approximation of* $L^2(\mathsf{R}$

The

Proof: The moments of Φ are given in terms of the Fourier transform by the formula

$$\int_{-\infty}^{\infty} x^m \Phi(x)\, dx = (-i)^m \hat{\Phi}^{(m)}(0)$$

here $\hat{\Phi}^{(m)}$ denotes the mth derivative of $\hat{\Phi}$. From (2.15) it follo s that $\hat{\Phi}(0) = 1$ since $\hat{\phi}(0) = 1$. By differentiating both sides of (2.16) e obtain

$$2^m \hat{\Phi}^{(m)}(2\xi) = \sum_{n=0}^{m} \binom{m}{n} \hat{\Phi}^{(m-n)}(\xi) M_0^{(n)}(\xi)\,.$$

When $\xi = 0$, e have

$$\hat{\Phi}^{(m)}(0) = \frac{1}{2^m - 1} \sum_{n=1}^{m} \binom{m}{n} \hat{\Phi}^{(m-n)}(0) M_0^{(n)}(0)\,. \tag{2.20}$$

Differentiating (2.18), e obtain

$M_0^{(n)}($

In order to compute the moments of the scaling function, it is not necessary to evaluate the defining integral,

$$\mu_m = \int_{-\infty}^{\infty} x^m \phi(x)\, dx \,.$$

Instead, e have the follo ing recursive formula

$$\begin{aligned} \mu_0 &= 1\,, \\ \mu_m &= \frac{1}{2^m - 1} \sum_{n=1}^{m} \binom{m}{n} \nu_n \mu_{m-n}\,, \quad m \geq 1 \end{aligned} \tag{2.22}$$

here

$$\nu_n = \frac{1}{\sqrt{2}} \sum_l l^n h_l \,.$$

The numbers $\{\nu_n\}$ are the (normalized) moments of the sequence $\{h_l\}$, and are easily computed hen $\{h_l\}$ is of finite length.

The formula (2.22) is ell-kno n and is derived as follo s. Using the t o-scale difference equation (2.8), e have

$$\begin{aligned} \mu_m &= \int x^m \phi(x)\, dx \\ &= \sqrt{2} \sum_l h_l \int x^m \phi(2x - l)\, dx \\ &= \frac{1}{\sqrt{2}} \sum_l h_l \int \left(\frac{x + l}{2}\right)^m \phi(x)\, dx \\ &= \frac{1}{2^m} \sum_{n=0}^{m} \binom{m}{n} \frac{1}{\sqrt{2}} \sum_l l^n h_l \int x^{m-n} \phi(x)\, dx \\ &= \frac{1}{2^m} \sum_{n=0}^{m} \binom{m}{n} \nu_n \mu_{m-n}\,, \end{aligned}$$

from hich (2.22) easil **l** o ff io Ssing e havh) **y** P P d ⊂ P P

Thus, the t o-scale difference equation for Φ may be expressed in terms of the variable x as

$$\Phi(x) = \frac{1}{2} \sum_m a_m \Phi(2x - m)\,. \tag{2.25}$$

Setting $x = n$, $n \in \mathbb{Z}$ e have

$$\Phi(n) = \frac{1}{2} \sum_m a_m \Phi(2n - m)\,.$$

The interpolating property (2.14) implies that $a_{2n} = 2\delta_{n,0}\,,$ here δ denotes the Kronecker delta. Using (2.24) it is easy to sho that $a_{-m} = a_m$, and e use these observations to rite

$$\Phi(x) = \Phi(2x) + \frac{1}{2} \sum_{m \geq 1} a_{2m-1} \left[\Phi(2x - 2m + 1) + \Phi(2x - 1 + 2m)\right]\,. \tag{2.26}$$

If Φ is compactly supported, as it must be if ϕ has compact support, then e understand that only finitely many of the coefficients in (2.26) are non-zero.

2.1.4 Examples

Perhaps the simplest example of an MRA is one hose elements are piece- ise constant on dyadic intervals. The scaling function for this MRA is the characteristic function of the interval $[0, 1)$,

$$\phi(x) = \begin{cases} 1\,, & 0 \leq x < 1 \\ 0\,, & \text{other ise}\,. \end{cases} \tag{2.27}$$

For historical reasons this is kno n as the Haar system. The Fourier transform is

$$\begin{aligned} \hat{\phi}(\xi) &= \int_0^1 e^{ix\xi}\, dx = \frac{e^{i\xi} - 1}{i\xi} \\ &= \left(\frac{e^{i\xi/2} - 1}{i\xi/2}\right) \left(\frac{e^{i\xi/2} + 1}{2}\right) \\ &= \hat{\phi}(\xi/2) m_0(\xi/2)\,. \end{aligned}$$

From this expression, e can read off the trigonometric polynomial m_0, i.e.

$$m_0(\xi) = \left(\frac{1 + e^{i\xi}}{2}\right) = \frac{1}{\sqrt{2}} \sum_l h_l e^{il\xi}\,,$$

hich implies that $h_0 = h_1 = 1/\sqrt{2}$, and $h_l = 0$ other ise. Furthermore, from the explicit form of $m_0(\xi)$, e see that this function has a single zero at $\xi = \pi$.

The t o-scale difference equation satisfied by (2.27) is

$$\phi(x) = \phi(2x) + \phi(2x-1)\,.$$

The autocorrelation of (2.27) is

$$\Phi(x) = \begin{cases} 1+x\,, & -1 \le x < 0 \\ 1-x\,, & 0 \le x < 1 \\ 0\,, & \text{other ise.} \end{cases}$$

It is easily verified that

$$\int_{-\infty}^{\infty} \Phi(x)\,dx = 1\,, \qquad \int_{-\infty}^{\infty} x\Phi(x)\,dx = 0\,.$$

Example(1) is the lo est order member of both families of scaling functions mentioned in this thesis, namely the central B-splines and the orthonormal scaling functions ith compact support constructed by Daubechies (see [7] or [8]). Spline spaces ill be described in more detail in Section 2.4. Daubechies scaling functions satisfy the t o-scale difference equation

$$\phi(x) = \sqrt{2}\sum_{l=0}^{2M-1} h_l \phi(2x-l)\,, \qquad \text{here} \quad \sum_{l=0}^{2M-1} h_l^2 = 1\,.$$

This scaling function provides an orthonormal basis for the subspaces of an MRA ith M vanishing moments, for $M = 1, 2, \ldots,$ here M is the multiplicity of the zero at $\xi = \pi$ in (2.12). The support of ϕ is the ith a

These scaling functions possess good approximation properties, but are difficult to evaluate point *for* $0 \le m \le M-1$.

Due to our assumption of compact support or exponential decay at infinity, the numbers μ_m are ell-defined for every integer $m \ge 0$. Before giving the proof of the proposition, e state and prove a series of lemmas.

Lemma 2.2.1 *If $\phi(x)$ is compactly supported or satisfies (2.31), then for each non-negative integer m there exists a constant C_m such that*

$$\sum_{k=-\infty}^{\infty} |x-k|^m |\phi(x-k)| \le C_m \tag{2.33}$$

for every real x, and the sum converges uniformly.

Proof: First assume that $\phi(x)$ is compactly supported. No

$$\sum_{k=-\infty}^{\infty} |x-k|^m |\phi(x-k)| = \lim_{N\to\infty} \sum_{k=-N}^{N} |x-k|^m |\phi(x-k)|\,,$$

provided that this limit exists. If the limit exists, then the limit function must be 1-periodic, so it is sufficient to consider $0 \le x < 1$. Let $[a,b]$ be the smallest closed interval that contains the support of $\phi(x)$. Then $\phi(x-k) = 0$ if $x-k > b$ or $x-k < a$, so it follo s that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} |x-k|^m |\phi(x-k)| = \sum_{k=k_0}^{k_1} |x-k|^m |\phi(x-k)|\,,$$

here $k_0 = -\lfloor b \rfloor$, $k_1 = -\lfloor a \rfloor$, and $\lfloor\cdot\rfloor$ denotes the greatest integer less than or equal to $(\cdot)$. Thus the sequence of partial sums converges for each x, and as the sum involves only a finite number of terms it converges uniformly. For $0 \le x < 1$, e have $|k| \le |x-k| \le |k|+1$, so that

$$\sum_{k=k_0}^{k_1} |x-k|^m |\phi(x-k)| \le \|\phi\|_\infty \sum_{k=k_0}^{k_1} (1+|k|)^m\,, \tag{2.34}$$

here $\|\phi\|_\infty = \sup|\phi(x)|, a \le x \le b$. It follo s that the sum is uniformly bounded by the constant on the right-hand side of (2.34).

Next assume that $\phi(x)$ has exponential decay at infinity, i.e. $|\phi(x)| \le Ae^{-\ |x|}$, for some positive constants A and α. No

$$\sum_{k=-\infty}^{\infty} |x-k|^m |\phi(x-k)| = \lim_{N\to\infty} \sum_{k=-N}^{N} |x-k|^m |\phi(x-k)|\,,$$

provided that this limit exists, and as before it is sufficient to consider $0 \le x < 1$. Then $|k| \le |x - k| \le |k| + 1$ implies that

$$|x-k|^m |\phi(x-k)| \le A|x-k|^m e^{- \ |x-k|} \le A(1+|k|)^m e^{- \ |k|} .$$

Thus,

$$\sum_{k=-N}^{N} |x-k|^m |\phi(x-k)| \le A \sum_{k=-N}^{N} (1+|k|)^m e^{- \ |k|} , \tag{2.35}$$

and as $N \to \infty$ the sequence of partial sums on the right converges for any non-negative integer m. It follo s that the sequence of partial sums on the left-hand side of (2.35) converges uniformly for each x, and the limit function is uniformly bounded by the constant $A \sum_{-\infty}^{\infty} (1+|k|)^m e^{- \ |k|}$. □

Lemma 2.2.2 *If $\phi(x)$ is compactly supported or satisfies (2.31), then $x^m \phi(x)$ is in $L^1(\mathbf{R})$ for each non-negative integer m, and we have the estimate*

$$\int_{-\infty}^{\infty} |x|^m |\phi(x)| \, dx \le C_m , \tag{2.36}$$

where the constants in (2.33) and (2.36) are identical A here ^

No assume that $\hat{\phi}^{(n)}(2l\pi) = 0$ for all non-zero integers l and for $0 \leq n \leq m-1$. Differentiate (2 and in general e have

$$\sum_{-\infty}^{\infty} k^m \phi(x-k) = \sum_{n=0}^{m} \binom{m}{n} x^{m-n} (-1)^n \mu_n \,, \qquad 0 \le m \le M-1 \,. \qquad (2$$

If $f \in L^2(\mathbf{R})$, then $\sum |s_k^j|^2 < \infty$. Since the basis $\phi_k^j, k \in \mathbb{Z}$ is orthonormal, it follo s that

$$\|P_j f\|^2 = \sum_{-\infty}^{\infty} |s_k^j|^2 .$$

We may also allo f to be a generalized function [3]. In this context, e take $\phi_k^j, k \in \mathbb{Z}$ to be the test functions, and V_j to be the space of test functions. Then f is a continuous linear functional that assigns a unique real number (f, ϕ_k^j) to each $\phi_k^j \in V_j$. If f i

We point out that in the inequality (2.46), in all practical cases the scale parameter j satisfies $j \leq 0$. Before proving the proposition e state and prove a combinatorial lemma.

Lemma 2.2.5 *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be arbitrary sequences of length M, then we have*

$$\sum_{n=0}^{M-1} a_n \sum_{m=0}^{M-1-n} \binom{n+m}{n} b_m c_{n+m} = \sum_{m=0}^{M-1} c_m \sum_{n=0}^{m} \binom{m}{n} a_n b_{m-n} \,. \tag{2.47}$$

Proof: For convenience denote the left-hand side of (2.47) by l.h.s. Expanding the summation over n e have

$$\text{l.h.s.} = a_0 \sum_{m=0}^{M-1} \binom{0+m}{0} b_m c_{0+m} + \cdots + a_{M-1} \sum_{m=0}^{0} \binom{M-1+m}{M-1} b_m c_{M-1+m} \,.$$

No regroup these terms, factoring out in turn c_0, c_1, etc. Since c_0 appears only in the first term, c_1 appears only in the first and second terms, etc., e obtain

$$\text{l.h.s.} = c_0 \left\{ \binom{0}{0} a_0 b_0 \right\} + c_1 \left\{ \binom{1}{0} a_0 b_1 + \binom{1}{1} a_1 b_0 \right\}$$

$$+ \cdots + c_{M-1} \left\{ \binom{M-1}{0} a_0 b_{M-1} + \binom{M-1}{1} a_1 b_{M-2} + \cdots + \binom{M-1}{M-1} a_{M-1} b_0 \right\} .$$

This can evidently be ritten in compact notation as

$$\text{l.h.s.} = \sum_{m=0}^{M-1} c_m \sum_{n=0}^{m} \binom{m}{n} a_m b_{m-n} \,,$$

hich verifies (2.47). □

Proof of the Proposition: We first obtain an expression for the coefficient s_k^j in terms of the derivatives of f. Expanding f in a Taylor series about $(2^j k)$ e have

$$f(x) = \sum_{m=0}^{M-1} \frac{f^{(m)}(2^j k)}{m!} (x - 2^j k)^m + \frac{f^{(M)}(\xi_k)}{M!} (x - 2^j k)^M$$

here ξ_k lies bet een x and $(2^j k)$. Using this expression e have

$$\begin{aligned} s_k^j &= 2^{-j/2} \int f(x) \phi(2^{-j} x - k)\, dx \\ &= 2^{-j/2} \sum_{m=0}^{M-1} \frac{f^{(m)}(2^j k)}{m!} 2^{mj} \int (2^{-j} x - k)^m \phi(2^{-j} x - k)\, dx + 2^{j/2} \varepsilon_k^j \\ &= 2^{j/2} \sum_{m=0}^{M-1} \frac{f^{(m)}(2^j k)}{m!} 2^{mj} \mu_m + 2^{j/2} \varepsilon_k^j \end{aligned}$$

so that (2.52) reduces to

$$(P_j f)(x) - E_j(x) = f(x)\,.$$

This proves (2.45).

No let us examine the error terms. Clearly

$$|E_j^1(x)| \le C_0 \sup_k |\varepsilon_k^j|\,,$$

here e have used Lemma(2.2.1). No consider

$$\begin{aligned} |\varepsilon_k^j| &\le 2^{Mj} \int \frac{|f^{(M)}(\xi_k)|}{M!} |2^{-j}x - k|^M |\phi(2^{-j}x - k)|\, dx \\ &\le 2^{(M+1)j} \sup_{\xi \in I_j(x)} \frac{|f^{(M)}(\xi)|}{M!} \int |x|^M |\phi(x)|\, dx \\ &\le 2^{(M+1)j} C_M \sup_{\xi \in I_j(x)} \frac{|f^{(M)}(\xi)|}{M!} \end{aligned}$$

here e have used Lemma(2.2.2). Thus e have

$$|E_j^1(x)| \le 2^{(M+1)j} C_0 C_M \sup_{\xi \in I_j(x)} \frac{|f^{(M)}(\xi)|}{M!} \tag{2.53}$$

Similarly e have

$$|E_j^2(x)| \le 2^{Mj} C' C_M \sup_{\xi \in I_j(x)} \frac{|f^{(M)}(\xi)|}{M!}$$

here $C' = \sum_{m=0}^{M-1} (|\mu_m|/m!) 2^{mj}$. This proves (2.46). □

2.3 Multiresolution Approximation of Kernels

2.3.1 Kernels on V_j

In general, a kernel on V_j is an expression of the form

$$T_j(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} t_{m,n}^j \phi_m^j(x) \phi_n^j(y)\,,$$

here the function $\phi(x) \in V_0$ is the scaling function. In this thesis, e approximate kernels of the form $K(x, y) = K(x - y)$, and in this case the coefficients satisfy

$$t_{m,n}^j = t_{m-n}^j\,. \tag{2.54}$$

Therefore, for our purposes it is sufficient to vie a kernel as an expression of the form

$$T_j(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} t^j_{m-n} \phi^j_m(x) \phi^j_n(y) \,,$$

here the sequence $\{t^j_n\}$ belongs to $l^2(\mathbb{Z})$. Thus, to build an approximation to a given kernel $K(x-y)$ on the subspace V_j, it is necessary only to compute the appropriate coefficients. Conversely, the kernel T_j is completely determined once the coefficients $\{t^j_n\}$ are kno n.

We note that (2.54) does not imply that $T_j(x,y) = T_j(x-y)$, i.e. T_j is not a convolutional kernel. Ho ever, e do have the identity

$$T_j(x + 2^j k, y + 2^j k) = T_j(x,y) \,,$$

hich sho s that T_j is a "block convolution". To state this another ay, T_j is a periodic function of period 2^j [illegible]

Ho ever, e also have

$$
\begin{aligned}
(P_j K P_j f)(x) &= \int_{-\infty}^{\infty} T_j(x,y) f(y)\, dy \\
&= \int \sum_m \sum_n t^j_{m-n} \phi^j_m(x) \phi^j_n(y) f(y)\, dy \\
&= \sum_{m=-\infty}^{\infty} \phi^j_m(x) \sum_{n=-\infty}^{\infty} t^j_{m-n} \int \phi^j_n(y) f(y)\, dy \\
&= \sum_{m=-\infty}^{\infty} \phi^j_m(x) \sum_{n=-\infty}^{\infty} s^j_n t^j_{m-n}\,.
\end{aligned}
$$

Equating these t o expressions, e obtain

$$t^j_{m-n} = (K\phi^j_n, \phi^j_m) = \int\int K(x-y)\phi^j_m(x)\phi^j_n(y)\, dy\, dx\,. \tag{2.55}$$

Using a change of variables and reversing the order of integration e can re rite (2.55) as

$$t^j_n = \int_{-\infty}^{\infty} K(x)\Phi(2^{-j}x - n)\, dx\,, \tag{2.56}$$

here Φ is the autocorrelation introduced in Section 2.1.3. Note also the operator identity

$$T_j = P_j K P_j\,.$$

The follo ing proposition establishes a bound on the difference bet een a kernel K and its multiresolution approximation T_j.

Proposition 2.3.1 *Let ϕ be a compactly supported scaling function in an MRA with M vanishing moments. Let T_j denote the projection of a kernel K onto the subspace V_j, : i i n i 2 t i fia i n n i l*

Suppose that K *is at least* M *times* c

No use Proposition(2.2.1) to re rite (2.62) as

$$\begin{aligned} T_j(x,y) - E_j(x,y) &= \sum_{m=0}^{M-1} \frac{K^{(m)}(x-y)}{m!}(-1)^m 2^{mj} \sum_{n=0}^{m} \binom{m}{n} (-1)^n \mu_{m-n}\mu_n \\ &= K(x-y) \end{aligned}$$

having also used Lemma(2.1.1), and Proposition(2.1.1). This proves (2.58).

No consider the error terms. Taking $m = M$ in (5.39), e have

$$\sup_{(k,l)\in\mathcal{K}\times\mathcal{L}} |\varepsilon^j_{k-l}| \le 2^{(M+1)j} C_M \sup_{(\xi,\eta)\in R_j(x,y)} \frac{|K^{(M)}(\xi-\eta)|}{M!}.$$

Using this inequality, e have

$$\begin{aligned} & |E_j(x,y)| \\ \le\; & \sup_{(k,l)\in\mathcal{K}\times\mathcal{L}} |\varepsilon^j_{k-l}| \left(\sum_k |\phi(2^{-j}x-k)| \right) \left(\sum_l |\phi(2^{-j}y-l)| \right) \\ & + 2^{Mj} \sup_{(\xi,\eta)\in R_j(x,y)} \frac{|K^{(M)}(\xi-\eta)|}{M!} \sum_{n=0}^{M} \binom{M}{n} \\ & \times \left(\sum_k |2^{-j}x-k|^{M-n} |\phi(2^{-j}x-k)| \right) \left(\sum_l |2^{-j}y-l|^{n} |\phi(2^{-j}y-l)| \right) \\ \le\; & 2^{Mj} \sup_{(\xi,\eta)\in R_j(x,y)} \frac{|K^{(M)}(\xi-\eta)|}{M!} \left(C_1^2 + \sum_{n=0}^{M} \binom{M}{n} C_{M-n} C_n \right), \end{aligned}$$

hich proves (2.59). The constants are those provided by Lemma(2.2.1). □

2.3.3 Trigonometric Expansion of a Kernel

The follo ing theorem provides an efficient method for computing the value of $T_j(x,y)$ at a given point (x,y). As stated above (Section 2.3.1), the kernel $T_j(x,y)$ is not a function of the difference $(x-y)$ alone, a fact hich makes it difficult to tabulate. Ho ever, it turns out that T_j may be represented by a sum of functions that depend only on $(x-y)$, and being functions of a single variable, are easily tabulated. The proof of this exploits the fact that T_j is periodic on a fixed diagonal $x - y = \textit{constant}$.

The resulting series expansion (2.64) converges rapidly, and in our implementation e retain terms only for $|n| \le 3$. This high rate of convergence follo s from the rapid decay of the Fourier transform of the scaling function.

This result does not require the scaling function ϕ to belong to an orthonormal system.

Theorem 2.3.1 *Let $\phi(x)$ be a scaling function for the subspace V_0, which is continuous and has a piecewise continuous derivative. Let $T_j(x,y)$ be a kernel on V_j, which has the form*

$$T_j(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} t^j_{m-n} \phi^j_m(x) \phi^j_n(y)\,, \tag{2.63}$$

where $\sum |t^j_k|^2 < \infty$. Then we have the identity,

$$2^j T_j(2^j x, 2^j y) = \sum_{n=-\infty}^{\infty} e^{in\pi(x+y)} I_n(x-y)\,, \tag{2.64}$$

where

$$I_n(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iz\xi} \hat{t}^j(\xi - n\pi) \hat{\Phi}_n(\xi)\, d\xi\,, \tag{2.65}$$

$$\hat{t}^j(\xi) = \sum_{-\infty}^{\infty} t^j_k e^{ik\xi}\,, \tag{2.66}$$

and

$$\hat{\Phi}_n(\xi) = \hat{\phi}(\xi - n\pi)\overline{\hat{\phi}}(\xi + n\pi)\,. \tag{2.67}$$

Furthermore, for each $(x,y) \in \mathbf{R}^2$, the right-hand-side of (2.64) converges uniformly to $2^j T_j(2^j x, 2^j y)$.

Example: In order to illustrate the decay of the functions $\hat{\Phi}_n(\xi)$, assume that $\phi(x)$ is the central B-spline of degree $(2m-1)$. Then e have

$$\hat{\Phi}_n(\xi) = \left[\frac{\sin\left(\frac{\xi}{2} + \frac{n\pi}{2}\right)}{\frac{\xi}{2} + \frac{n\pi}{2}}\right]^{2m} \left[\frac{\sin\left(\frac{\xi}{2} - \frac{n\pi}{2}\right)}{\frac{\xi}{2} - \frac{n\pi}{2}}\right]^{2m}$$

 hich obviously decays rapidly as $|n| \to \infty$.

Proof of the Theorem:

Alternatively, e can rite

$$2^j T_j(2^j x, 2^j y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi(x-y)} \hat{t}^j(\xi) \left| \hat{\phi}(\xi) \right|^2 d\xi$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\xi(x-y)} t^j(\xi - n\pi) \mathrm{Re}\left\{ e^{in\pi(x+y)} \hat{\Phi}_n(\xi) \right\} d\xi \,. \tag{2.73}$$

When the functions $\hat{\Phi}_n(\xi)$ are real, this reduces to

$$2^j T_j(2^j x, 2^j y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi(x-y)} \hat{t}^j(\xi) \left| \hat{\phi}(\xi) \right|^2 d\xi$$

$$+ \sum_{n=1}^{\infty} \cos[n\pi(x+y)] \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\xi(x-y)} t^j(\xi - n\pi) \hat{\Phi}_n(\xi) \, d\xi \,. \tag{2.74}$$

2.4 Spline MRAs

An important class of multiresolution analyses use splines as a starting point (see e.g. [5] or [20]). In this thesis, e restrict our attention to the central B-splines of odd degree, though splines of even degree could also be used. Integer translates of these functions form a Riesz basis.

We denote by $\beta^{(M-1)}(x)$ the spline that is piece ise polynomial of degree $(M-1)$, for $M = 2, 4, \ldots$. By ay of example, the lo est order case, corresponding to $M = 2$, is the ell-kno n "hat function",

$$\beta^{(1)}(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| \geq 1 \,. \end{cases} \tag{2.75}$$

These functions are compactly su **g** f **E** d, **f** A **d** f **B** here $\beta^{(0)}(x)$ is the characteristic function of the interval $[-1/2, 1/2)$. (Here e allo m to be any non-negative integer.) We can also rite

$$\beta^{(m)}(x) = \int_{-1/2}^{1/2} \beta^{(m-1)}(x-t)\, dt\,, \quad m \geq 1\,.$$

Using the fact that $\beta^{(m)}$ is an $(m+1)$-fold convolut

2.4.1 The Battle-Lemarié Scaling Function

The scaling function for the orthonormal system in a spline MRA is kno n as the Battle-Lemarié scaling function (see e.g. [8, pp.146-152]). This scaling function is the result of applying the orthogonalization procedure (2.7) to the B-spline. The Battle-Lemarié function is thus defined in the Fourier domain by the equation

$$M$$

$$\hat{\phi}^{(M-1)}(\xi)$$ [illegible]

Using Poisson's summation formula, the relationship (2

Let M be an even, positive integer and let $(M-1)$ be the degree of the B-spline β. Construct t o sets of numbers $\{q_m\}_{m=0}^{M-1}$ and $\{Q_m\}_{m=0}^{2M-1}$ according to the formulae

$$\sum_{n=0}^{m} \binom{m}{n} (-1)^n q_{m-n} \mu_n = \begin{cases} 1\,, & m = 0 \\ 0\,, & m > 0 \end{cases} \qquad \text{for } 0 \leq m \leq M-1 \tag{2.97}$$

and

$$\sum_{n=0}^{m} \binom{m}{n} (-1)^n Q_{m-n} \mathcal{M}_n = \begin{cases} 1\,, & m = 0 \\ 0\,, & m > 0 \end{cases} \qquad \text{for } 0 \leq m \leq 2M -$$

where

$$|E_j(x)| \le 2^{Mj} C \sup_{\xi \in I_j(x)} \frac{|f^{(M)}(\xi)|}{M!} \,. \tag{2.103}$$

The constant C depends on β but not on f.

Proof: Let x_0 be an arbitrary real number. Expand f in a Taylor series about x_0 to obtain the expression for the mth derivative,

$$f^{(m)}(x) = \sum_{n=0}^{M-1-m} \frac{f^{(n+m)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(M)}(\xi_m)}{M!}(x-x_0)^M \,,$$

here $0 \le m \le M-1$, and ξ_m lies bet een x and x_0. Set $x = 2^j k$ to obtain

$$f^{(m)}(2^j k) = \sum_{n=0}^{M-1-m} \frac{f^{(n+m)}(x_0)}{n!}(2^j k - x_0)^n + \frac{f^{(M)}(\xi_{m,k})}{M!}(2^j k - x_0)^M \,,$$

here $\xi_{m,k}$ lies bet een x_0 and $(2^j k)$ for each m. No use this expression together ith (2.99) to rite

$$\begin{aligned}
(P_j f)(x_0) &= 2^{-j/2} \sum_k s_k^j \beta(2^{-j}x_0 - k) \\
&= \sum_k \beta(2^{-j}x_0 - k) \sum_{m=0}^{M-1} \frac{f^{(m)}(2^j k)}{m!} \, 2^{mj} q_m \\
&= \sum_k \beta(2^{-j}x_0 - k) \sum_{m=0}^{M-1} \frac{q_m}{m!} \left\{ \sum_{n=0}^{M-1-m} \frac{f^{(n+m)}(x_0)}{n!}(2^j k - x_0)^n \right. \\
&\quad \left. + \frac{f^{(M)}(\xi_{m,k})}{M!}(2^j k - x_0)^M \right\} \\
&= \sum_{m=0}^{M-1} q_m \sum_{n=0}^{M-1-m} \frac{f^{(n+m)}(x_0)}{m!n!} 2^{(m+n)j}(-1)^n \\
&\quad \times \sum_k (2^{-j}x_0 - k)^n \beta(2^{-j}x_0 - k) + E_j(x_0) \,,
\end{aligned} \tag{2.104}$$

here e have put

$$E_j(x) = \sum_{m=0}^{M-1} \frac{q_m}{m!} \sum_k \frac{f^{(M)}(\xi_{m,k})}{M!}(2^j k - x)^M \beta(2^{-j}x - k) \,. \tag{2.105}$$

Continuing from (2.104), and using the identity (2.80), e have

$$\begin{aligned}
(P_j f)(x_0) - E_j(x_0) &= \sum_{m=0}^{M-1} q_m \sum_{n=0}^{M-1-m} \frac{f^{(n+m)}(x_0)}{m!n!} 2^{(m+n)j}(-1)^n \mu_n \\
&= \sum_{m=0}^{M-1} q_m \sum_{n=0}^{M-1-m} \binom{n+m}{m} \frac{f^{(n+m)}(x_0)}{(n+m)!} 2^{(m+n)j}(-1)^n \mu_n \,.
\end{aligned}$$

No use (2.47) to transform the right-hand side of this equation, thus obtaining

$$\begin{aligned}(P_j f)(x_0) - E_j(x_0) &= \sum_{m=0}^{M-1} \frac{f^{(m)}(x_0)}{m!} 2^{mj} \sum_{n=0}^{m} \binom{m}{n} (-1)^n q_{m-n} \mu_n \\ &= f(x_0)\,,\end{aligned}$$

here e have used (2.97). This proves (2.102).

Finally, e observe that, since β is compactly supported, there is a constant C_M such that

$$\sum_{-\infty}^{\infty} |x-k|^M |\beta(x-k)| \le C_M\,,$$

by Lemma(2.2.1). Thus, from (2.105) e have

$$\begin{aligned}|E_j(x)| &\le \left(\sum_{m=0}^{M-1} \frac{|q_m|}{m!} 2^{mj} \right) 2^{Mj} \sum_k |2^{-j}x - k|^M |\beta(2^{-j}x-k)| \frac{|f^{(M)}(\xi_{m,k})|}{M!} \\ &\le 2^{Mj} C' C_M \sup_{\xi \in I_j(x)} \frac{|f^{(M)}(\xi)|}{M!}\,,\end{aligned}$$

here $C' = \sum_{m=0}^{M-1} 2^{mj}(|q_m|/m!)$. This verifies (2.103). □

Proposition 2.4.2 *Let β be the central B-spline of degree $(M-1)$, where M is an even positive integer. Let $T_j = P_j K P_j$ denote the projection of a kernel K onto the subspace V_j, given by*

$$T_j(x,y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} t^j_{k-l} \beta^j_k(x) \beta^j_l(y) \qquad (2.106)$$

where the coefficients are given by (2.100). For a given point $(x,y) \in \mathbb{R}^2$, let $R_j(x,y)$ be the rectangle formed by the union of the supports of all basis functions which are non-zero at (x,y). Thus, if

$$I_j(x) = \bigcup_{k \in \mathcal{K}} \mathrm{supp}\,(\beta^j_k)\,, \qquad \mathcal{K} = \{k \in \mathbb{Z} | \beta^j_k(x) \neq 0\}$$

and

$$I_j(y) = \bigcup_{l \in \mathcal{L}} \mathrm{supp}\,(\beta^j_l)\,, \qquad \mathcal{L} = \{l \in \mathbb{Z} | \beta^j_l(y) \neq 0\}$$

then

$$R_j(x,y) = I_j(x) \times I_j(y)\,.$$

Suppose that K is at least M times continuously differentiable on $R_j(x,y)$. Then we have

$$T_j(x,y) = K(x-y) + E_j(x,y)\,, \qquad (2.107)$$

where

$$|E_j(x,y)| \leq 2^{Mj} C \sup_{(\xi,\eta)\in R_j(x,y)} \frac{|K^{(M)}(\xi-\eta)|}{(M)!}\,. \tag{2.108}$$

The constant C depends on β but not on K.

The proof of this proposition is similar t hich follo s from (2.92). Comparison of this expression ith (2.81) sho s that

$$\hat{A}(\xi) = |\hat{\phi}(\xi)|^2$$

here ϕ is the Battle-Lemarié scaling function. Thus the correlation function A is equal to the autocorrelation of the Battle-Lemarié scaling function almost every here, hich by Proposition(2.1.1) must have vanishing moments. Let $\mathcal{A}_m$ denote the mth moment of A. Then e have

$$\mathcal{A}_m = \begin{cases} 1\,, & m = 0 \\ 0\,, & m > 0 \end{cases}$$

for $0 \leq m \leq 2M - 1$, and as in Lemma(2.1.1) e have

$$\mathcal{A}_m = \sum_{n=0}^{m} \binom{m}{n} (-1)^n q_{m-n} \mu_n \,.$$

Since the moments of the B-spline $\{\mu_n\}$ are easy to compute, (use the recursive formula in equation 2.22) e can use the above t o equations to obtain $\{q_m\}_{m=0}^{2M-1}$.

hapter 3

The Fast Summation Algorithm in One Dimension

In this chapter e describe our approach for problems in one dimension. Our approach in higher dimensions is similar. Indeed, to develop an algorithm in t o dimensions, the only additional machinery used is singular value decomposition of the coefficient matrix, but this ill be discussed in Chapter(4).

3.1 General Description

1. Our goal is to compute the numbers $\{g_m\}$, here

$$g_m = \sum_{\substack{n=1 \\ n \neq m}}^{N} K(x_m - x_n) f_n \,, \qquad 1 \leq m \leq N \,. \tag{3.1}$$

We assume that the kernel K is singular on the line $x - y = 0$, i.e.

$$|K(x-y)| \to \infty \quad \text{as} \quad (x-y) \to 0 \,,$$

but is at least M times continuously differentiable on any region that does not contain the line $x = y$. Moreover, e assume that there is a "band idth" B such that the Mth derivative is uniformly bounded outside the band $|x - y| < B$, i.e.

$$|K^{(M)}(x-y)| \leq C \quad \text{for } |x-y| \geq B. \tag{3.2}$$

2. We choose a level of refinement ($j \leq 0$), a multiresolution analysis ith M vanishing moments, and construct the projection of the kernel onto the subspace V_j, given b f

As discussed in Chapter(2) this construction requires only the computation of the coefficients $\{t^j_n\}$, given by

$$t^j_n = \int_{-\infty}^{\infty} K(x)\Phi(2^{-j}x - n)\,dx\,, \quad n \in \mathbb{Z}$$

here Φ is the autocorrelation of the scaling function ϕ. This can be done ahead of time and the coefficients $\{t^j_n\}$ are then stored in memory. This portion of the computation is part of the initialization.

3. Using Proposition(2.3.1) together ith (3.2) e have the estimate

$$|K(x-y) - T_j(x,y)| \leq \epsilon \quad \text{for} \quad |x-y| \geq 2^j B\,, \tag{3.4}$$

for any $\epsilon > 0$, provided that M and $j \leq 0$ have been chosen so that $C(2^{Mj}/M!) < \epsilon$. (There is some abuse of notation here, since the constants C and B are not necessarily identical to the co and

$$s_l^j = \sum_{n=1}^{N} f_n \phi_l^j(x_n)\,. \tag{3.9}$$

small, and hence points (

2. The number of vanishing moments M corresponds to a B-spline of degree $(2m-1)$. More precisely, e have $M-1 = 2m-1$, so that M must be an even, positive integer. Once this parameter and the level of refinement $j \leq 0$ have been chosen, the projection of the kernel K is

$$T_j(x,y) = \sum_{k=0}^{J-1} \sum_{l=0}^{J-1} t^j_{k-l} \beta^j_k(x) \beta^j_l(y)\,, \tag{3.15}$$

here $J = 2^{-j}$, and β denotes the B-spline of degree $(2m-1)$. The coefficients $\{t^j_n\}$ for $|n| \leq (J-1)$ are computed using the formula (2.100).

3. Note that, in anticipation ion

6. Having obtained the coefficients (3.18) e can no evaluate the expansion

$$g^j(x) = \quad {}^{J-1}$$

Step	Procedure	Complexit

and then e

N	T_{LF}	T_{HF}	T_{tot}	T_{dir}	E_2	E_∞
64	0.0036	0.0037	0.0073	0.0077	0.46860E-05	0.35852E-05
128	0.0066	0.0079	0.0145	0.0278	0.41383E-05	0.47034E-05
256	0.0138	0.0155	0.0293	0.1044	0.38780E-05	0.42643E-05
512	0.0264	0.0300	0.0564	0.4042	0.36356E-05	0.43642E-05
1024	0.0552	0.0591	0.1143	1.5946	0.33201E-05	0.33570E-05
2048	0.1089	0.1175	0.2264	6.3151	0.31335E-05	0.33471E-05
4096	0.2379	0.2351	0.4731	25.2104	0.33210E-05	0.31553E-05

Th

N	T_{LF}	T_{HF}	T_{tot}	T_{dir}	E_2	E_∞
64	0.0138	0.0130	0.0268	0.0086	0.50929E-14	0.26622E-14
128	0.0261	0.0260	0.0521	0.0318	0.23669E-14	0.16785E-14
256	0.0525	0.0477	0.1002	0.1244	0.43058E-14	0.18982E-14
512	0.1111	0.0925	0.2036	0.4910	0.85681E-14	0.43438E-14
1024	0.2344	0.1770	0.4114	2.0113	0.78821E-14	0.25543E-14
2048	0.5288	0.3593	0.8881	8.0242	0.91355E-14	0.27125E-14
4096	1.0738	0.7384	1.8122	32.1545	0.11375E-13	0.66191E-14

Table 3.3: Implementation in one dimension using B-splines of degree 11, break-even at about 220 particles.

can then be computed ith the standard FFT. In order to determine the coefficients $\{g_m\}$, e invert (3.31) to get

$$g_m = \frac{1}{N}\sum_{j=0}^{N-1} \hat{f}_j e^{-2\pi i m j/N} \,.$$

Next substitute expression (3.29) for $\hat{f}_j$ to get

$$\begin{aligned} g_m &= \frac{1}{N}\sum_{j=0}^{N-1} e^{-2\pi i m j/N} \sum_{n=0}^{N-1} f_n\, e^{2\pi i j x_n} \\ &= \sum_{n=0}^{N-1} f_n \frac{1}{N}\sum_{j=0}^{N-1} e^{-2\pi i j (m/N - x_n)} \,, \end{aligned} \tag{3.32}$$

here $f_n = f(x_n)$. No e have

$$\begin{aligned} \sum_{j=0}^{N-1} e^{-2\pi i j (m/N - x_n)} &= \frac{1 - e^{-2\pi i (m - N x_n)}}{1 - e^{-2\pi i (m/N - x_n)}} \\ &= -2e^{i\pi N x_n} \sin \end{aligned}$$

(x x ss Nessx x x l s ss (s2 t ss k

No , in order to find an explicit representation for $\tilde{T}_j$, e proceed as follo s. Assume that e have constructed T_j and T_{j+1}. Then e have

$$T_j(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} t^j_{m-n} \phi^j_m(x) \phi^j_n(y) \, , \tag{3.41}$$

and

$$T_{j+1}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} t^{j+1}_{m-n} \phi^{j+1}_m(x) \phi^{j+1}_n(y) \, . \tag{3.42}$$

In order to express T_{j+1} in terms of the basis functions ϕ^j_n, $n \in \mathbb{Z}$ in V_j, e use the t o-scale difference equation (2.8) satisfied by the scaling function $\phi(x)$, hich takes the general form

$$\phi^{j+1}_k(x) = \sum_l h_{l-2k} \phi^j_l(x) \, . \tag{3.43}$$

Using (3.43) in (3.42), e have

$$T_{j+1}(x,y) \quad = \quad \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}$$

$V_j, \ldots, V_{j+n}$. This is done as follo s. Recall that (see equation (3.9)) the coefficients $\{s_l^j\}$ that represent the particle distribution on V_j are given by

s

hapter 4

The Fast Summation Algorithm in Higher Dimensions

In this chapter, e explain ho to extend our one-dimensional scheme to higher dimensions. The algorithm has been implemented in t o dimensions, and e give details of this and also present some numerical results. As mentioned above, there is little here beyond the use of vector notation that is not completely analogous to the one-dimensional case, except for singular value decomposition of the coefficient matrix (this matrix is defi

4.2 The Fast Summation Algorithm in Two Dimensions

1. Our goal is to compute

$$g(x_m, y_m) = \quad {}^{N}$$

2.3.3. Since e are using central B-splines as basis functions, the functions $\hat{\Phi}_n(\xi)$ are real. In this case, e can rite

$$
\begin{aligned}
& 4^j\, T_j(2^j x, 2^j y, 2^j x', 2^j y') \\
= \;& I_{0,0}(z, z') + 2\sum_{n=1}^{\infty} \cos(n\pi w') I_{0,n}(z, z') + 2\sum_{m=1}^{\infty} \cos(m\pi w) I_{m,0}(z, z') \;\; y, 2
\end{aligned}
$$

g u tt lt f

Defining

$$
\begin{aligned}
U_m^{(r)}(z) &= \sum_k u_k^{(r)} \Phi_m(z-k) \\
V_n^{(r)}(z') &= \sum_l v_l^{(r)} \Phi_n(z'-l)\,,
\end{aligned}
$$

e can express $I_{m,n}$ in the form

$$
I_{m,n}(z,z') = \sum_{r=1}^{R} \sigma_r U_m^{(r)}(z) V_n^{(r)}(z')\,. \tag{4.13}
$$

5. Substituting (4.13) into (4.8), e obtain our final expression for the trigonometric expansion of the kernel,

$$
\begin{aligned}
4^j\, T_j(2^j x, 2^j y, 2^j x', 2^j y') &= \sum_{r=1}^{R} \sigma_r \left\{ U_0^{(r)}(z) + 2\sum_{m=1}^{\infty} \cos(m\pi w) U_m^{(r)}(z) \right\} \\
&\times \left\{ V_0^{(r)}(z') + 2\sum_{n=1}^{\infty} \cos(n\pi w') V_n^{(r)}(z') \right\}\,.
\end{aligned} \tag{4.14}
$$

Once the functions $U_m^{(r)}(z)$ and $V_n^{(r)}(z')$ have been tabulated, the cost of eval-

The errors are computed according to the follo ing formulae. Suppose that $\mathbf{x}$ is the "exact" N-length vector, and $\tilde{\mathbf{x}}$ is the approximation. Then e compute the relative errors

$$E = \frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|}{\|\mathbf{x}\|},$$

here

$$\|\mathbf{x}\|_2^2 = \frac{1}{N}\sum_{i=1}^{N} |x_i|^2, \qquad \|\mathbf{x}\|_\infty = \max_{1\le i\le N} |x_i|.$$

The timings ere done on 10/30/96 using a Sun Sparc-20 orkstation.

N	T_{LF}	T_{HF}	T_{tot}	T_{dir}	E_2	E_∞
64	0.0920	0.0435	0.1355	0.0124		

hapter 5

Regularization of Singular Operators

One often encounters discrete sums that do not have a direct analogue as an integral operator. For example, the expression

$$(Kf)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x-y|^2}\, dy \tag{5.1}$$

requires special interpretation as an operator. By contrast, the discrete sum

$$g(x_m) = \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{f(x_n)}{|x_m - x_n|^2}\,, \quad 1 \leq m \leq N \tag{5.2}$$

presents no difficulties. In our approach to the computation of such sums, e construct kernels $T_j : V_j \to V_j\,, j \in \mathbb{Z}$ hich are approximations of the kernel functions that appear in the summation problem. Ho ever, it is clear that the same approximation T_j may be used to provide a definition of the corresponding integral operator.

Thus, for example, if e have constructed a kernel T_j such that the sum

$$g^j(x_m) = \sum_{\substack{n=1 \\ n\neq m}}^{N} T_j(x_m, x_n) f(x_n)\,, \quad 1 \leq m \leq N \tag{5.3}$$

5.1 Preliminary Considerations

In order to construct a kernel T_j that represents a linear operator K on the subspace V_j of a multiresolution analysis, it is

For operators satisfying these conditions, the projections of the corresponding kernels $T_j : V_j \to V_j$ have the form

$$T_j(x, y) = 2^{-j(1+\ )} T(2^{-j}x, 2^{-j}y)\,, \tag{5.12}$$

for any $j \in \mathbb{Z}$, here

$$T(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} t_{m-n} \phi(x - m)\phi(y - n)\,, \tag{5.13}$$

and the coefficients are given by the functional

$$t_n = (\phi_n^0\,, K\phi_0^0)\,, \tag{5.14}$$

here

$$(\phi_n^0\,, K\phi_0^0) = \int_{-\infty}^{\infty} \phi(x - n)(K\phi)(x)\, dx \tag{5.15}$$

henever the right-hand side of (5.15) exists.

As examples, e consider belo the follo ing integral operators ith algebraic singularities

$$(Kf)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{(x - y)^{1+\ }}\, dy\,, \quad \alpha \geq -1 \tag{5.16}$$

and

$$(Kf)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x - y|^{1+\ }}\, dy\,, \quad \alpha \geq -1 \tag{5.17}$$

For $\alpha = 0$, the expression (5.15) defines a bounded operator on $L^2(\mathbf{R})$, provided that e consider the principle value at $x = y$. When suitably scaled, this example is kno n as the Hilbert transform. Ho ever, for most values of α, the operators (5.16) and (5.17) are not bounded on $L^2(\mathbf{R})$. It has been sho n in [2] that the derivative operators (d/dx) , for $\alpha = 1, 2, \ldots$ also satisfy the conditions stated above, and have representations of the form (5.12).

5.2 Classical Regularization of Divergent Integrals

Our task is to give a meaning to the functional (5.14) hen the integral (5.15) is divergent. To this end, e first recall ho such problems have been addressed up to no .

As an example, consider the expression

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{x}\, dx\,, \tag{5.18}$$

here ϕ is an infinitely differentiable function ith compact support, $\phi \in C_0^\infty(\mathbb{R})$. This integral converges for all test functions ϕ that vanish in a neighborhood of the origin. Ho ever, the integral diverges if $\phi(0) \neq 0$. We no ask hether it is possible to define a functional (x^{-1}, ϕ) such that, for all test functions ϕ that vanish in a neighborhood of zero, the functional has the value given by (5.18). The functional (x^{-1}, ϕ) is then called a regularization of the divergent integral (5.18) (see e.g. [13, pp.10-12]).

The functional

$$(x^{-1}, \phi) = \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x}\, dx + \int_{-\epsilon}^{\epsilon} \frac{\phi(x) - \phi(0)}{x}\, dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x}\, dx\,, \tag{5.19}$$

here ϵ is any positive real number, obviously reduces to (5.18) if $\phi(0) = 0$. Moreover, since $\phi(x) - \phi(0) \sim x\phi'(0)$ as $x \to 0$, it

In the follo ing section, e present a method for regularization of the integral (5.15), thus providing a meaning to the functional (5.14), hich utilizes the multiresolution approach. For integral operators ith algebraic singularities, our approach produces the same results as the classical regularization method, ith the additional benefit of a practical scheme for computation. Furthermore, it appears that our approach may also be applied to classes of operators ith more general types of singularities.

5.3 A Multiresolution Approach to Regularization

5.3.1 Choice of the Scaling Function

Throughout this chapter, e limit our consideration to a specific family of MRAs, namely, those belonging to the compactly supported scaling functions constructed by I. Daubechies [7]. These scaling functions and their autocorrelations ere introduced in Section 2.1.4.

We recall that for the MRA ith M vanishing moments, the autocorrelation $\Phi(x)$ is supported on the interval $[1-2M, 2M-1]$, and also that the coefficients $\{a_m\}$ in the t o-scale difference equation (2.28) for $\Phi(x)$ are given explicitly in (2.29).

5.3.2 Two-Scale Difference Equation for the Coefficients

In this section e derive the follo ing necessary condition on the functional (5.14),

$$2^{-} \; t_n = t_{2n} + \frac{1}{2} \sum_{m=1}^{M} a_{2m-1} \left[t_{2n-2m+1} + t_{2n-1+2m} \right] . \qquad (5.24)$$

We refer to (5.24) as the t o-scale difference equation. This equation is the tool that e i obtained by setting $j = 0$ in (5.11). The scaling function $\phi(x)$ satisfies the t o-scale difference equation

$$\phi(x) = \sqrt{2} \sum_{k=0}^{2M-1} h_k \phi(2x - k) \,,$$

hich e re rite as

$$\phi_n^{(0)}(x) =$$

for the regularization of such operators. We sho its consistency ith the classical definition (see Section 5

We are no in a position to give a constructive definition of the multiresolution regularization of the operator K.

Definition 5

5.3.4 Classical vs Multiresolution Regularization

Consider the integral operator

$$(Kf)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{(x-y)^{1+}}\, dy \,,$$

here $\alpha = 0, 1, 2, \ldots$ as an example. Projection of this operator onto the subspace V_0 of the MRA ith M vanishing moments (see Section 5.3.1) requires evaluation of the integral

$$t_n = \int_{-\infty}^{\infty} \phi(x-n)(K\phi)(x)\, dx \,,$$

hich can also be ritten as

$$t_n = \int_{-\infty}^{\infty} \frac{\Phi(x-n)}{x^{1+}}\, dx \,, \tag{5.31}$$

here $\Phi(x)$ is the autocorrelation of $\phi(x)$ (see equation (2.56)). The support of $\Phi(x)$ is the closed interval $[1-2M, 2M-1]$. Hence the integral (5.31) is ell defined henever $|n| \geq 2M$, since for these values of n, the support of $\Phi(x-n)$ does not contain the origin.

We consider a regularizatioegn rnyrd 2

and the regularization of this integral is the functional

$$t_n = \left(x^{-1-} , \Phi(x - n\right.$$

No , since $a_{-m} = a_m$, it follo s that

$$\int_0^\infty \frac{dx}{x^{1+}} \frac{1}{2} \sum_m a_m \left\{ \Phi(x - 2n - m) - \Phi(x - 2n + m) \right\} = 0\,.$$

Therefore, e have

$$\begin{aligned} 2^{-} \; t_n &= \frac{1}{2} \sum_m a_m \int_0^\infty \frac{dx}{x^{1+}} [\Phi(x - 2n + m) - \Phi(x + 2n - m) \\ &\quad - 2 \sum_{k=0}^{(\;\; -1)/2} \frac{\Phi^{(2k)}(2n - m)}{(2k)!} x^{2k}] \\ &= \frac{1}{2} \sum_m a_m t_{2n-m} = \frac{1}{2} \sum_m a_m t_{2n+m}\,. \end{aligned}$$

As this result is valid for all $n \in \mathbb{Z}$, the t o-scale difference equation (5.24) is satisfied. The procedure for (5.35) is identical. □

Proposition 5.3.1 *The classical regularization and the multiresolution regularization of (5.31) produce identical results.*

Proof: The coefficients t_n for $|n| \geq 2M$ computed in Steps 1 and 2 of the algorithm of Section 5.3 agree ith the integral expression (5.31)

It is sho n in [2] that eigenvectors of A belonging to the eigenvalues 2^{-} are composed of the values of the αth derivative of $\Phi(x)$, taken at the integers. That is, if

$$v = \{\Phi^{(\ )}(1-2M), \ldots, \Phi^{(\ )}(2M-1)\},$$

then $Av = 2^{-}\ v$. Furthermore, the αth derivative $\Phi^{(\ )}(x)$ exists and is continuous if and only if 2^{-} is an eigenvalue of A, and is nondegenerate.

Hence, to prove the proposition, it is necessary and sufficient to sho that

$$\sum_{1-2M}^{2M-1} t_n \, \Phi^{(\ )}(n) = 0 \,, \tag{5.36}$$

here the coefficients t_n are given by (5.33) or (5.35).

No consider the follo ing. Since $\Phi(-x) = \Phi(x)$, it follo s that if α is even, then $\Phi^{(\ )}(-x) = \Phi^{(\ )}(x)$, and if α is odd, then $\Phi^{(\ )}(-x) = -\Phi^{(\ )}(x)$. No , from (5.31) it follo s that if α is even, then $t_{-n} = -t_n$, and if α is odd, then $t_{-n} = t_n$. Thus (5.36) is al ays satisfied. □

Remark: The fact that the classical regularization and the multiresolution regularization of (5.31) produce identical results proves that the limit of the regularized kernel T_j, as $j \to -\infty$, is independent of the basis chosen, at least in those cases in hich both methods are applicable.

Example: Consider $\alpha = 1$ in (5.33). The regularization is

$$t_n = \int_0^\infty \frac{dx}{x^2} \{\Phi(x-n) + \Phi(x+n) - 2\Phi(n)\} \,.$$

Since $t_{-n} = t_n$, it is sufficient to consider only $n \geq 0$. Also, since $\Phi(0) = 1$, and $\Phi(n) = 0$ if $n \neq 0$, e have

$$\begin{aligned} t_0 &= 2\int_0^\infty \frac{\Phi(x)-1}{x^2}\, dx \\ t_n &= \int_0^\infty \frac{\Phi(x-n)+\Phi(x+n)}{x^2}\, dx \,, \quad n \geq 1 \,. \end{aligned} \tag{5.37}$$

We note that, in (5.37), e must choose $\Phi(x)$ that belongs to an MRA ith $M \geq 3$, since for $M = 1$ or 2 the function $\Phi(x)$ does not have a second derivative, and the integrals in (5.37) do not converge.

To convince the reader that the classical approach (i.e. equations (5.37)) produces the same results as the algorithm of Section 5.3.3, e have computed the coefficients t_n, for $0 \leq n \leq 5$ for the MRA ith $M = 3$. We list the values in the table belo , rounded to three digits. The column marked "Classical" is obtained by evaluating (5.37) using quadrature formulae, and has only t o digits of accuracy. By contrast, the column marked "MRA" is correct to the accuracy ϵ chosen in Step 1 of the algorithm.

n	Classical	MRA
0	-5.471	-5.508
1	2.359	2.375
2	-0.058	-0.059
3	0.150	0.152
4	0.064	0.064
5	0.040	0.041

5.3.5 Asymptotic Condition for Integral Operators

For integral operators, the coefficients $t_n, n \in \mathbb{Z}$ are computed according to the formula

$$t_n^j = \int_{-\infty}^{\infty} K(x)\Phi(2^{-j}x - n)\, dx \,,$$

here $\Phi(x)$

With the change of variable $y = 2^{-j}x - n$ this reduces to

$$
\begin{aligned}
t_n^j &= \sum_{l=0}^{m-1} \frac{K^{(l)}(2^j n)}{l!} 2^{(l+1)j} \int_{-\infty}^{\infty} y^l \Phi(y)\, dy + \varepsilon_n^j \\
&= 2^j K(2^j n) + \varepsilon_n^j ,
\end{aligned}
$$

hich follo s from Proposition(2.1.1), since $m - 1 \leq 2M - 1$.) 2

No let us consider the error term (5.4my rm r hor e tegrc d rm s mj

=

here $t_n, n \in \mathbb{Z}$ are the coefficients that satisfy the t o-scale difference equation (5.24). Recall the definition of the trigonometric series $M_0(\xi)$ from equation (2.17),

$$M_0(\xi) = \frac{1}{4} \sum_{-\infty}^{\infty} a_m e^{im\xi} ,$$

hich for the MRAs under consideration (see Section 5.3.1) has the form

$$M_0(\xi) \quad = \quad \frac{1}{2} + \frac{1}{4} \sum_{m=1}^{M} a_{2m-1}$$

But $\sup|M_0(\xi)|^2 = 1$, so e have

$$\|M_0^* g\|^2 \leq 4\|g\|^2\,,$$

hich implies that $\|M_0^*\| \leq 2$.

Next, e make use of the fact that $M_0(0) = 1$. Define the sequence of functions $g_n(\xi) = \sqrt{n}$ if $|\xi| < 1/n, 0$ other ise. Then e have

$$\begin{aligned} \|M_0^* g_n\|^2 &= \int_{-\pi}^{\pi} |(M_0^* g_n)(\xi)|^2 \, d\xi \\ &= n \int_{-1/2n}^{1/2n} |2M_0(\xi)|^2 \, d\xi\,, \end{aligned}$$

and by the continuity of M_0 e have

$$\|M_0^* g_n\|^2 \to |2M_0(0)|^2 = 4\,,$$

Proposition 5.4.3 *Let K be a linear, homogeneous operator that commutes with translation. If the quantities*

$$t_n = \int_{-\infty}^{\infty} \phi(x-n)(K\phi)(x)\,dx$$

are well-defined for every integer n, then the series $\hat{t}(\xi) = \sum t_n e^{in\xi}$ satisfies the eigenvalue equation

$$(M_0\hat{t})(\xi) = 2^{-} \quad \hat{t}(\xi)\,.$$

Thus, the sequence t_n , $n \in \mathbb{Z}$ may be identified ith a trigonometric series $\hat{t}(\xi)$, hich is an eigenvector of the operator M_0. The question therefore arises hether or not the regularization produced by the algorithm of Section 5.3.3 is an eigenvector of M_0.

Proposition 5.4.4 *Let K be a linear operator that is homogeneous of degree α and commutes with translation. Let $t_n, n \in \mathbb{Z}$ be the regularization computed according to the algorithm of Section 5.3.3. Let $\hat{t}(\xi)$ be the formal trigonometric series $\sum t_n e^{in\xi}$. If 2^{-} is not an eigenvalue of the matrix A, then $\hat{t}$ satisfies the eigenvalue equation*

$$(M_0\hat{t})(\xi) = 2^{-} \quad \hat{t}(\xi)\,.$$

If 2^{-} is an eigenvalue of A, then $\hat{t}$ satisfies the equation

$$(M_0\hat{t})(\xi) = 2^{-} \quad \hat{t}(\xi) + r(\xi)\,, \tag{5.48}$$

where $r(\xi)$ is a trigonometric polynomial, and $(M_0 r)(\xi) = 2^{-} \; r(\xi)$. Furthermore, we have

$$r(\xi) = \sum_{1-2M}^{2M-1} r_n e^{in\xi}\,,$$

where the vector $\mathbf{r} = \{r_{1-2M}\,, \ldots\,, r_{2M-1}\}$ is given by

$$\mathbf{b} - (\lambda I - A)\tau = \mathbf{r}\,.$$

Thus, if the system (5.28) has a solution, then $\mathbf{r} = 0$, and in place of (5.48) we have

$$(M_0\hat{t})(\xi) = 2^{-} \quad \hat{t}(\xi)\,.$$

2

Proof: Let us define the tri metric series

Then $\hat{t}(\xi) = \sigma(\xi) + \tau(\xi)$. We also define the trigonometric polynomial

$$b(\xi) = \sum_{1-2M}^{2M-1} b_n e^{in\xi} \,,$$

here $\mathbf{b} = \{b_{1-2M},$

Bibliography

[1] C. Anderson. A method of local corrections for computing the velocity field due to a distribution of vortex blobs. *Journal of Computational Physics*, 62, 1986.

[2] G. Beylkin. On the representation

[12] P. E ald. Die berechnung optischer und elektrostatischer gitterpotentiale. *Annalen der Physik*, 64:253–287

Appendix A

Solution of the System $\lambda x = Ax + b$

In this appendix e consider solution of a linear system

$$\lambda x = Ax + b\,, \tag{A.1}$$

here A is a square, diagonalizable matrix, but is other ise arbitrary. We assume that λ is an eigenvalue of A (other ise there is no difficulty). When A is symmetric, the situation is ell understood.

In the symmetric case, invariant subspaces belonging to distinct eigenvalues are orthogonal. If a symmetric matrix A has m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, here $m \leq n$, and if P_i is the projector onto the invariant subspace belonging to λ_i, then e have the operator identity

$$I = P_1 + \cdots + P_m\,,$$

called a partition of unity. Moreover, the system

$$\lambda_i x = Ax + b \tag{A.2}$$

has a solution if and only if $P_i b = 0$, and the solution is unique up to addition of an eigenvector of A belonging to λ_i.

It ill be sho n belo that, even if A is not symmetric, a sequence of matrices D_i can be constructed that satisfies

$$I = D_1 + \cdots + D_m\,,$$

the system (A.2) has a solution if and only if $D_i b = 0$, and the solution is unique up to addition of an eigenvector of A belonging to λ_i.

The matrices D_i imitate the projection matrices P_i in that they are idempotent ($D_i^2 = D_i$), and satisfy $D_i x = x \Leftrightarrow x = P_i x$. Ho ever, they are not symmetric unless A is symmetric, in hich case $D_i = P_i$.

In this appendix, e prove the follo ing proposition.

Proposition A.0.5 *Let A be a diagonalizable $(n \times n)$ matrix. Let* $\lambda_1, \ldots, \lambda$

No set

$$D = R(L^T R)^{-1} L^T , \tag{A.10}$$

here the linear independence of the columns of R

N

since by Lemma(A.0.3) $D_i v_j = 0$ if $j \neq i$ and $D_i v_i = v_i$. It follo s that (A.14) may be ritten as

$$x = D_1 x + \cdots + D_n x\,,$$

hich implies the identity (A.5).

We no investigate the solution of (A.6). First note that Lemma(A.0.4) implies the existence of the inverse in (A.7). Write

$$r = b - (\lambda_i I - A)x\,. \tag{A.15}$$

No , x is a solution to (A.6) if and only if $r = 0$ in (A.15). Substituting the expression (A.7) for x, equation (A.15) becomes

$$\begin{aligned} r &= b - (\lambda_i I - A)x \\ &= b - (\lambda_i I - A)S_i(\lambda_i I - S_i^T A S_i) \end{aligned}$$

Appendix B

Explicit Expression for $B_{m,n}(x)$

In this appendix e derive explicit expressions for the functions

$$B_{m,n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \hat{\beta}^{(m)}(\xi - n\pi) \hat{\beta}^{(m)}(\xi + n\pi)\, d\xi\,. \tag{B.1}$$

These are (Fourier transforms of) the functions $\hat{\Phi}_n(\xi)$ introduced in Section 2.3.3, equation (2.67), in connection ith the trigonometric expansion of a kernel. See also equation (3.23). Equation (B.1) corresponds to Φ_n hen the basis function is the central B-spline $\beta^{(m)}(x)$.

It ill be more convenient for the derivation to use the B-spline $N_m(x)$, in place of $\beta^{(m)}(x)$. This notation is used by some authors (e.g. [5]) to denote the spline hich is piece ise polynomial of degree $(m-1)$, and is supported on the interval $[0, m]$. The relationship bet een $N_m(x)$ and $\beta^{(m)}(x)$ is

$$\beta^{(m-1)}(x) = N_m(x + m/2)\,.$$

It follo s from this relationship that

$$\hat{N}_m(\xi - n\pi)\overline{\hat{N}}_m(\xi + n\pi) = \hat{\beta}^{(m-1)}(\xi - n\pi)\hat{\beta}^{(m-1)}(\xi + n\pi)\,.$$

Thus, it is sufficient to prove the follo ing proposition.

Proposition B.0.6 *For $m \geq 1$, let $N_m(x)$ denote the spline function which is piecewise polynomial of degree $(m-1)$, and which vanishes outside the interval $[0, m]$. Define*

$$B_{m-1,n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \hat{N}_m(\xi - n\pi)\overline{\hat{N}}_m(\xi + n\pi)\, d\xi\,. \tag{B.2}$$

Then, for $n \geq 1$, we have

$$B_{m-1,n}(x) = \sum_{k=0}^{m-1} \frac{(-i)^{m-k} a_{m,k}}{(2n\pi)^{m+k}} \left[e^{in\pi x} + (-1)^{m-k} e^{-in\pi x}\right] \left(\frac{d}{dx}\right)^{m+k} N_{2m}(x+m)\,, \tag{B.3}$$

where

$$a_{m,k} = \frac{(m-1+k)!}{(m-1)!\,k!}\,. \tag{B.4}$$

Note that the spline $N_{2m}(x+m)$ is equal to the central B-spline of degree $(2m-1)$.

Example: Take $m = 4$. The spline $N_4(x+2)$ is equal to the central B-spline $\beta^{(3)}(x)$. When $n = 0$, it is obvious that

$$B_{4,0}(x) = N_8(x+4) = \beta^{(7)}(x)\,,$$

hich is the autocorrelation of $N_4(x)$. If $n \geq 1$, then using the formula (B.3), e have $B_{3,n}(x) =$

$$\left\{\frac{\cos(n\pi x)}{8(n\pi)^4}\frac{d^4}{dx^4} - \frac{\sin(n\pi x)}{4(n\pi)^5}\frac{d^5}{dx^5} - \frac{5\cos(n\pi x)}{16(n\pi)^6}\frac{d^6}{dx^6} + \frac{5\sin(n\pi x)}{16(n\pi)^7}\frac{d^7}{dx^7}\right\}\beta^{(7)}(x)\,.$$

Explicit expressions for the various derivatives of the central B-spline of degree seven are listed in the follo ing table. We only consider $x \geq 0$ since the central B-splines are even functions. Note also that the last column is piece ise constant, thus the seventh derivative does not exist at integer points, but since it is multiplied by the term $\sin(n\pi x)$, hich is zero at integer points, this does not cause any discontinuity in the function

To prove the lemma, e use induction. First, consider

$$\begin{aligned}(e^{-i\xi} - 2 + e^{i\xi})e^{-ix\xi} &= e^{-i\xi(x+1)} - 2e^{-i\xi x} + e^{-i\xi(x-1)} \\ &= \triangle^2 e^{-i\xi(x+1)}\,.\end{aligned}$$

(In hat follo s, the algebra is some hat tedious but entirely straightfor ard, and for the most part has been omitted.) No , assume that (B.5) holds, Similarly, expanding $R_m(z)$ about $z = -1$, e have

$$\begin{aligned} R_m(z) &= \frac{f(z)}{(z+1)^m} \\ &= \frac{1}{(z+1)^m} \sum_{k=0}^{\infty} \frac{f^{(k)}(-1)}{k!} z^k \\ &= (-1)^m \sum_{k=0}^{\infty} a_{m,k} (z+1)^{k-m} . \end{aligned}$$

The partial fraction decomposition is the sum of the singular parts of these t o expansions, hich consists of the first $(m-1)$ terms of each expansion. Combining these t o finite sums gives (B.6). □

In (B.8), e consider the "function" $F_m(x)$ as a formal expression only, hich ill be useful later on. We do not claim that the integral converges for all positive integers m.

Lemma B.0.8 *Define*

$$F_m(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{\xi^m} \, d\xi \, , \qquad m \geq 1 \, . \tag{B.8}$$

These functions are related to the B-splines through the equation

$$\triangle^m F_m(x) = (-i)^m N_m(x) \, , \quad \text{a.e.} \tag{B.9}$$

Proof: We first note that for $m = 1$, the integral is ell-kno n, and e have

$$F_1(x) = \frac{-i}{\pi} \int_0^{\infty} \frac{\sin(x\xi)}{\xi} \, d\xi = \frac{-i}{2}$$

With this in hand, assume that (B.9) holds for index $m-1$, and use (B.11) to compute

$$\begin{aligned}
\triangle^m F_m(x) &= \triangle^m \frac{-ix}{m-1} F_{m-1}(x) \\
&= \frac{-i}{m-1} \sum_{n=0}^{m} \binom{m}{n} (\triangle^n x)\, \triangle^{m-n} F_{m-1}(x-n) \\
&= \frac{-i}{m-1} \left\{ \binom{m}{0} x \triangle^m F_{m-1}(x) + \binom{m}{1} \triangle^{m-1} F_{m-1}(x-1) \right\}.
\end{aligned}$$

Here e have used the product formula for the back ard difference operator, as ell as the fact that $\triangle^n x = 0$, for $n \geq 2$. No e have

$$\begin{aligned}
\triangle^m F_m(x) &= \frac{-i}{m-1} \left\{ x \triangle^m F_{m-1}(x) + m \triangle^{m-1} F_{m-1}(x-1) \right\} \\
&= \frac{-ix}{m-1} \left[\triangle^{m-1} F_{m-1}(x) - \triangle^{m-1} F_{m-1}(x-1) \right] \\
&\quad - \frac{im}{m-1} \triangle^{m-1} F_{m-1}(x-1) \\
&= \frac{-i}{m-1} \left\{ x \triangle^{m-1} F_{m-1}(x) + (m-x) \triangle^{m-1} F_{m-1}(x-1) \right\} \\
&= (-i)^m \left\{ \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1) \right\} \\
&= (-i)^m N_m(x),
\end{aligned}$$

here e have used (B.12) and our induction hypothesis. □

Proof of Proposition: Making the change of variable $\xi \leftarrow \xi + n\pi$, and using the formula

$$\hat{N}_m(\xi) = \left(\hat{N}_1(\xi)\right)^m = \left(\frac{e^{i\xi} - 1}{i\xi}\right)^m,$$

e can re rite (B.2) as

$$\begin{aligned}
e^{in\pi x} B_{m-1,n}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \left(\frac{e^{i\xi} - 1}{i\xi}\right)^m \left(\frac{e^{-i\xi} - 1}{-i(\xi + 2n\pi)}\right)^m d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{\left(e^{-i\xi} - 2 + e^{i\xi}\right)^m}{[-\xi(\xi + 2n\pi)]^m} d\xi.
\end{aligned}$$

Using Lemma(B.0.6), this equality can be ritten as

$$\begin{aligned}
e^{in\pi x} B_{m-1,n}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\triangle^{2m} e^{-i\xi(x+m)}}{[-\xi(\xi + 2n\pi)]^m} d\xi \\
&= \triangle^{2m} G_{m,n}(x+m),
\end{aligned}$$

here e have put

$$G_{m,n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{[-\xi(\xi + 2n\pi)]^m} \, d\xi \, .$$

No , using Lemma(B.0.7), e can express the denominator of the integrand in the form

$$\begin{aligned} \frac{1}{[-\xi(\xi+2n\pi)]^m} &= \frac{(-1)^m}{(2n\pi)^m} R_m\left(\frac{\xi}{2n\pi}\right) \\ &= \sum_{k=0}^{m-1} \frac{a_{m,k}}{(2n\pi)^{m+k}} \left\{ \frac{1}{(\xi+2n\pi)^{m-k}} + \frac{(-1)^{m-k}}{\xi^{m-k}} \right\} . \end{aligned}$$

Using this equality, e can rite

$$G_{m,n}(x) = \sum_{k=0}^{m-1} \frac{a_{m,k}}{(2n\pi)^{m+k}} \left[e^{2\pi i n x} + (-1)^{m-k} \right] \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{\xi^{m-k}} \, d\xi \, .$$