1. Nonlinear equations: Solution:

The function $f(x) = x - g(x)$ is continuous on $[a, b]$ and crosses the axis: $f(a) = a - g(a) < 0 < b - g(b) = f(b)$. Hence, there exists at least one zero, u, of f (that is, a fixed point of g) in $[a, b]$. Assume also that $g(v) = v = u$. Then $0 < |u - v| = |g(u) - g(v)| < \lambda|u - v| < |u - v|$, a contradiction. Thus, $u = v$ and we have proved uniqueness. Convergence holds as follows:

$$|u - x_{n+1}| = |g(u) - g(x_n)| \leq \lambda|u - x_n|,$$

which, by induction, implies convergence of x_n to u according to

$$|u - x_n| \leq \lambda^n|u - x_0|.$$

The explicit linear convergence bound now follows:

$$|x_{n+1} - u| = |g(x_n) - g(u)| \leq \lambda|x_n - u|.$$

2. **Numerical quadrature:** **Solution:**

We first note that symmetry tells that $\alpha = \gamma$. (If there were solutions with $\alpha \neq \gamma$, we would obtain equally valid ones with α and β interchanged, and averaging these formulas will also create valid formulas with the coefficients for $u(0)$ and $u(1)$ equal.)

In all the three cases, the resulting formula should be exact for the test function $u(x) = 1$, implying

$$1 = 2\alpha + \beta. \tag{1}$$

It thus only remains in each of the three cases to find a second test function, giving a second equation for the two unknowns.

a. Trapezoidal rule:

This quadrature formula should be exact for piecewise linear functions. Hence, consider for example

$$u(x) = \begin{cases} x & , \quad 0 \le x \le \frac{1}{2} \\ 1 - x & , \quad \frac{1}{2} \le x \le 1 \end{cases} .$$

It should now hold $\int_0^1 u(x)\, dx = \frac{1}{4} = \alpha \cdot 0 + \beta \cdot \frac{1}{2} + \alpha \cdot 0$. Together with (1), we obtain $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$.

b. Simpson's formula:

This method should be exact for an arbitrary quadratic function, in particular for $u(x) = x(1-x)$. We now get $\int_0^1 u(x)\, dx = \frac{1}{6} = \alpha \cdot 0 + \beta \cdot \frac{1}{4} + \alpha \cdot 0$, i.e. $\alpha = \frac{1}{6}$, $\beta = \frac{2}{3}$.

c. Natural spline:

In this case, it is natural to construct a second test function as follows: Let $u(x)$ over $0 \le x \le \frac{1}{2}$ be a cubic polynomial with the properties

$$u(0) = 0, \;\; u''(0) = 0, \;\; u(\tfrac{1}{2}) \neq 0, \;\; u'(\tfrac{1}{2}) = 0, \tag{2}$$

and then define $u(x)$ for $\frac{1}{2} \le x \le 1$ as the reflection around $x = \frac{1}{2}$, i.e. as $u(1-x)$. This function $u(x)$ is a natural cubic spline over [0,1]. It is straightforward to see that for ex. $u(x) = x - \frac{4}{3}x^3$ obeys the requirements (2), and satisfies $u(\frac{1}{2}) = \frac{1}{3}$, $\int_0^{1/2} u(x)\, dx = \frac{5}{48}$. We thus obtain as our second equation $\frac{5}{24} = \frac{1}{3}\beta$, and can conclude that $\alpha = \frac{3}{16}$, $\beta = \frac{5}{8}$.

3. <u>Interpolation Approximation:</u> <u>Solution:</u>

Since e is continuous, there must exist $\alpha, \beta \quad [a, b]$ that satisfy

$$M = e(\alpha) = \max_{x \quad [a,b]} [illegible]$$

4. <u>Linear algebra:</u> <u>Solution:</u>

(a) This is a result of the following identities:

$$\max_{x=0} \frac{QARx^{\;2}}{x^{\;2}} = \max_{y=0} \frac{QARR\;y^{\;2}}{R\;y^{\;2}} = \max_{y=0} \frac{QAy^{\;2}}{y^{\;2}} = \max_{y=0} \frac{< A^T Q\;QAy, y >}{< y, y >} = \max_{y=0} \frac{< A^T Ay, y >}{< y, y >}.$$

(b) $A = U \Lambda V$, where U, V $^{n \times n}$ are unitary and Λ is $n \times n$ diagonal.

(c) $A = U \Lambda V = \Lambda = V \Lambda U = A = A^T$.

(d) Suppose $Au = \lambda u$, where $0 = u$ ^{n} and $|\lambda| = \rho$. Then $\rho(A) = \lambda u / u_{222} = {}_{2222} Au / u \leq A$.

(e) $A^{\;2} = max_{x=0} < A^T Ax, x > / < x, x >= max_{x=0} < A^2$ $=_{=0} < A^{222}$

6. **Numerical PDEs:** **Solution:**

a. The difference approximation is $\dfrac{u(x,t+k)-u(x,t)}{k}=\dfrac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}$.

b. Substitute $u(x,t)=\xi^{t/k}e^{i\omega x}$ into the difference approximation above to obtain $\xi=1+\frac{k}{h^2}2(\cos\omega h-1)$. When ωh varies over $[-\pi,$, the expression