

Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August 17, 2016 , 10 am – 1 pm.

Submit solutions to four (and no more) of the following six problems. Show all your work, and justify all your answers. No calculators allowed.

Problems and solutions given below:

1. Root finding / Nonlinear equations

Consider the scalar equation $F(x) = 0$. Assume α is a root of the equation.

- a. Give the recursion for the Newton method for approximating a root.
- b. Give conditions on $F(x)$ near

Solution: Nonlinear Equations

(a) Newton:

$$x_{j+1} \equiv x_j - \frac{F(x_j)}{F'(x_j)} \quad (1)$$

(c) Note that $F(\alpha) = 0$ and solve for α to get

$$\alpha - x = \frac{F(x) - F(\alpha)}{F'(x) - 2F'(\alpha)} (\alpha - x)^2.$$

$$\alpha - x_{j+1} = \frac{1}{2} \frac{F''(\alpha)}{F'(\alpha)} (\alpha - x_j)^2$$

$$|x_{j+1} - \alpha| \leq \frac{1}{2} \frac{F''(\alpha)}{F'(\alpha)} (\alpha - x_j)^2$$

(d) $M = 1/\alpha$, $\alpha = x_0$, $x_1 \leq x_0$, $x_2 \leq x_1$, \dots

$M|\alpha - x_1| < M|\alpha - x_0|$, $x_1 \in \mathcal{N}$, $x_2 \in \mathcal{N}$, \dots

x_1 as chosen so that $\alpha \neq x_0 \leq x_1$ and

$$M|\alpha - x_1| < (M|\alpha - x_0|)^2 / (\alpha - x_1) \leq (\alpha - x_0)^2 / (\alpha - x_1),$$

which implies $M|\alpha - x_1| < M|\alpha - x_0|$, and, thus, $x_1 \in \mathcal{N}$. By induction, this also implies $x_j \in \mathcal{N}$ for $j > 1$ and

with x_j

This $\alpha = x_0$ and convergence is quadratic.

2.

3. Interpolation / Approximation

- a. Define what is meant by *cubic splines* and, for these, *natural* and *not-a-knot* conditions.
- b. Determine the *not-a-knot* cubic spline $s(x)$ that satisfies the data $\begin{array}{c|cccc} x & 1 & 0 & 1 & 2 \\ \hline y & 2 & 3 & 4 & 1 \end{array}$.
- c. If, at the nodes $x = -h, 0, h$, one has function values y_{-h}, y_0, y_h and forms a quadratic interpolant $s(x)$, one obtains $s'(0) = [\frac{1}{2}y_{-h} - \frac{1}{2}y_h]/h$, i.e. the finite difference weights can be written as $[\frac{1}{2}, 0, -\frac{1}{2}]/h$. It might be tempting to replace the quadratic interpolant here with a natural cubic spline (hoping to increase the approximation's order of accuracy). Work out the weights you get in this case.

Solution:

- a. A *cubic spline* is a cubic polynomial between adjacent nodes, and features continuous function, first and second derivative at the nodes – i.e. the third derivative may be discontinuous at the nodes. Without additional end conditions, a cubic spline will have two free parameters. A *natural* cubic spline adds the two extra conditions that $s''(x) = 0$ at each end point. The *not-a-knot* cubic spline instead removes two 'freedoms', i.e. the cubic spline is not allowed to have a jump in its third derivative one node point in from each boundary.
- b. With four node points, and jumps in the third derivative not allowed at either of the two internal nodes, the spline becomes a single cubic, i.e. we can immediately find it, for ex., by Lagrange's or Newton's interpolation formulas. Choosing, for ex., the Newton approach, the divided difference table becomes

-1	-2			
0	-3	-1	0	
1	-4	-1	3	1
2	1	5		

from which we read off the polynomial as $s(x) = 2 - 1(x-1) - 0(x-1)x - 1(x-1)x(x-1) = x^3 - 2x^2 + 3x - 3$.

- c. Since the spline $s(x)$ is not discontinuous at $x = 0$ until in the third derivative, we can write it:

$$\begin{aligned} [h, 0] & \quad a + bx + cx^2 + dx^3 \\ [0, h] & \quad a + bx + cx^2 + ex^3 \end{aligned}$$

The natural end conditions give $2c - 6dh = 0$ and $2c - 6eh = 0$, resp., i.e. $e = d$. Enforcing the values at the nodes now give

$$\begin{aligned} a + bh + ch^2 + dh^3 &= y_{-h} \\ a &= y_0 \\ a + bh + ch^2 + dh^3 &= y_h \end{aligned}$$

Subtracting the top equation from the bottom one gives $2bh = y_h - y_{-h}$, and we obtain the same approximation for $s'(0)$

4. Linear Algebra

Consider the linear system $A\underline{x} = \underline{b}$, where $A_{n \times m}$, $\underline{x}_{m \times 1}$, $\underline{b}_{n \times 1}$.

a. Describe the three possible cases for existence and uniqueness of a solution of the linear system. Give criteria on A, \underline{b} that distinguish each case.

b. Let \underline{x}_{LS} be a minimizer of the least squares functional, that is, let

$$\|A\underline{x}_{LS} - \underline{b}\|_2 = \min_{\underline{x}} \|A\underline{x} - \underline{b}\|_2.$$

(i) Does \underline{x}_{LS} always exist? Explain your answer.

(ii) Give conditions on A, \underline{b} such that \underline{x}_{LS} is unique.

(iii) In the case of a unique solution, give an expression for the least squares solution \underline{x}_{LS} .

(iv) ITc[(4t43 -2.isol3(x)T32 Tc4r nu)-.695.e lean o]TTp resstoast squares solutioproe

Solution Linear Algebra

(a) The three cases are:

(i) If $\det(A) \neq 0$, there exists a unique solution. (ii) If $\det(A) = 0$ and the rank of A is less than the number of columns, there are no solutions. (iii) If $\det(A) = 0$ and the rank of A is equal to the number of columns, there exists an infinite number of solutions if:

(a) The rank of A is equal to the number of columns, there are no solutions. (b) The rank of A is less than the number of columns, there are infinite solutions.

(c) The rank of A is less than the number of columns, there are infinite solutions. (d) The rank of A is less than the number of columns, there are infinite solutions.

(e)

(f) $h \leq \text{Rank}$

(c) The pseudo inverse can be computed by using the singular value decomposition (SVD).

$$A = U\Sigma V^*$$

where $U_{n \times n}$ and $V_{m \times m}$ are unitary and $\Sigma_{n \times m}$ is diagonal:

$$\Sigma_{n \times m} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

and σ_j are the singular values. Then,

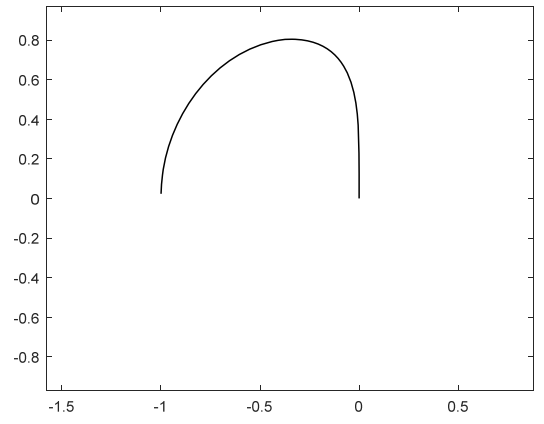
$$(A^*A)^\dagger A^* = A^\dagger = V\Sigma^\dagger U^*$$

where

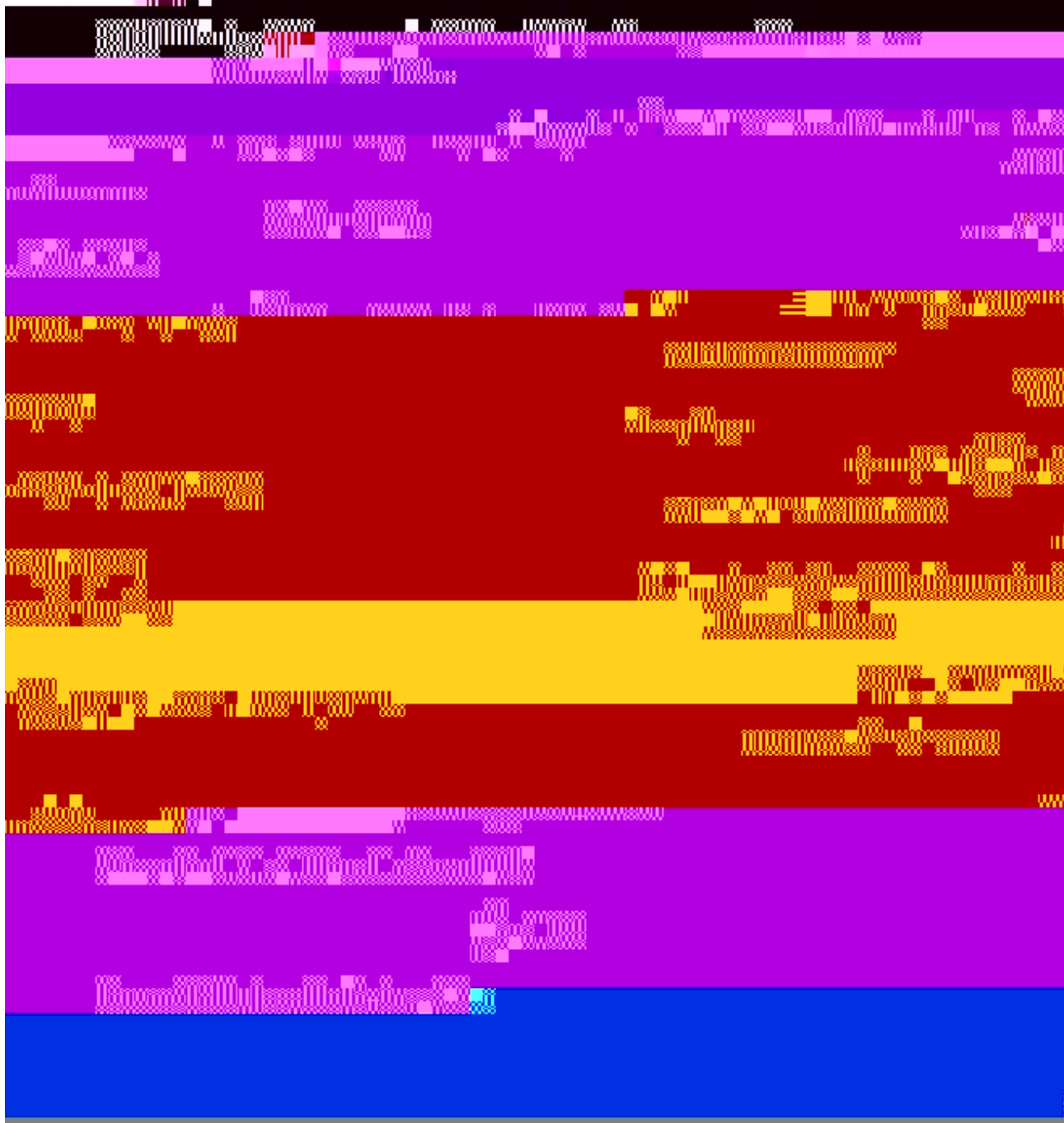
$$\Sigma_{m \times n}^\dagger = \text{diag}\{\dots, \sigma_j^\dagger, \dots\}$$

and

$$\sigma_j^\dagger = \begin{cases} \frac{1}{\sigma_j} & \text{if } \sigma_j > 0 \\ 0 & \text{if } \sigma_j = 0 \end{cases}$$



Solution PDE



$$\frac{d}{dt} \int_{\Omega} \rho \, dx = \int_{\Omega} \rho \, dx - \int_{\Omega} \rho \, dx = 0 \quad \text{for } t \in [0, \infty) \quad \text{with } \rho|_{t=0} = \rho_0$$

