1. **Nonlinear Equations** Given s alar e a ion f x 0

a es ri e Ne ons e od e an e od or a ro a in e sol ion.

- a e s ien ondi ions or Ne on and e an o onver e. sa is ed a a ra e ill ea onver e
- e e roo o onver en e or Ne on s e od.

d ri e Ne on s e od as a ed oin i era ion. a e s ien ondi ions or a eneral ed oin i era ion o onver e.

e ly e ri erion or ed oin i era ion o Ne on s e od and develo an al erna e roo or Ne on s e od.

Solution

a Ne on s e od Given x_0 le

$$x_{n+1} \quad x_n - \frac{f \; x_n}{f \;\; x_n}, \qquad n \quad 0.$$

e an e od Given x_0 x_1 le

$$x_{n+1} \quad x_n - f \; x_n \; \frac{x_n - x_{n-1}}{f \; x_n \; - f \; x_{n-1}} \qquad n \quad 1.$$

Ne on s e od e f α 0. ss e a ere e is s an in erval E $\alpha - \eta, \alpha$ η s a f x f x and f x are on in o s on E and

$$\frac{\text{a}_x \; f \;\; x}{2 \; \text{in}_x \; f \;\; x} \quad M,$$

and $\eta M < 1.0.$ en or any x_0 i E Ne on s e od ill onver e i ra e 2.0.

e an e od nder e s e ass ions i x_0 and x_1 are in E e e e an e od ill onver e i ra e $\frac{1+\sqrt{5}}{2} \sim 1.62$.

ee inson a es 5 , 60.

d e ne

$$g \; x \quad x - \frac{f \; x}{f \;\; x}.$$

Ne on s e od an e as as Given x

Numerical quadrature:

2. Assume that a quadrature rule, when discretizing with n nodes, possesses an error expansion of the form

$$I - I_n = \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \dots$$

Assume also that we, for a certain value of n, have numerically evaluated I_n , I_{2n} and I_{3n}.

a. Derive the best approximation that you can for the true value I of the integral.

b. The error in this approximation will be of the form $O(n^{-p})$ for a certain value of p. What is this value for p?

Solution:

a. With three numerically evaluated values, we can solve for three variables. For these we want to choose I, c_1 and c_2, at which point we only care about the obtained value for I. Abbreviating $\frac{c_1}{n} = d_1$ and $\frac{c_2}{n^2} = d_2$, we thus obtain the relations

$$\left\{ \begin{array}{lcl} I - I_n &=& d_1 + d_2 \\ I - I_{2n} &=& \frac{1}{2} d_1 + \frac{1}{4} d_2 \\ I - I_{3n} &=& \frac{1}{3} d_1 + \frac{1}{9} d_2 \end{array} \right. ,$$

or, written in the usual linear system form (separating 'knowns' from 'unknowns')

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -\frac{1}{2} & -\frac{1}{4} \\ 1 & -\frac{1}{3} & -\frac{1}{9} \end{bmatrix} \begin{bmatrix} I \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} I_n \\ I_{2n} \\ I_{3n} \end{bmatrix}$$

from which follows

$$I = \tfrac{1}{2}(I_n - 8I_{2n} + 9I_{3n}).$$

b. With the first two terms in the error expansion eliminated, it will continue from the third term and onwards (with modified coefficients), i.e. the error in the approximation above will be of the form $O(n^{-3})$.

Interpolation / Approximation:

3. The *General Hermite interpolation problem* amounts to finding a polynomial $p(x)$ of degree ${}_1 + {}_2 + \ldots + {}_n - 1$ that satisfies

$$
\begin{array}{ll}
p^{(i)}(x_1) = y_1^{(i)}, & i = 0, 1, \ldots, {}_1 - 1 \\
\quad\vdots & \qquad\vdots \\
p^{(i)}(x_n) = y_n^{(i)}, & i = 0, 1, \ldots, {}_n - 1,
\end{array}
$$

where the superscripts denotes derivatives, that is, we specify the first ${}_j - 1$ derivatives at the point x_j, for $j = 1, 2, \ldots, n$. Show that this problem has a unique solution whenever the x_i are distinct.

Hint: Set up the linear system for a small problem, recognize the pattern, and prove the general result.

Solution:

In all, there are ${}_1 + {}_2 + \ldots + {}_n = N$ conditions. Let the interpolation polynomial of degree $N - 1$ be $p(x) = {}_0 + {}_1 x + \ldots + {}_{N-1} x^{N-1}$. Each of the given conditions form one line in a linear system for the coefficients:

$$
\begin{bmatrix}
1 & x_1 & \pounds & \pounds & x_1^{N-1} \\
0 & 1 & \pounds & \pounds & (N-1)x_1^{N-2} \\
 & & \bullet & \pounds & \pounds \\
1 & x_2 & \pounds & \pounds & x_2^{N-1} \\
\S & \S & \S & \S & \S
\end{bmatrix}
\begin{bmatrix}
{}_0 \\ {}_1 \\ \S \\ \S \\ {}_{N-1}
\end{bmatrix}
=
\begin{bmatrix}
y_1^{(0)} \\ y_1^{(1)} \\ \S \\ \S \\ \S
\end{bmatrix}
$$

The task is to show that this $N \% N$ coefficient matrix is nonsingular, as this will imply both existence and uniqueness. One way to do this is to let the right hand side (RHS) be zero, and show that the problem then has only the zero solution.

With the RHS zero, the conditions that are imposed require $p(x)$ to have a zero of degree ${}_1$ at x_1, i.e. a factor $(x - x_1)^{{}_1}$; then likewise a factor of $(x - x_2)^{{}_2}$, etc. These required factors will imply that the polynomial $p(x)$ will have a total of N zeros (counting multiplicities). This is one above the actual degree of $p(x)$, implying that all the coefficients of $p(x)$ must be zero.

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4. Linear Algebra

Consider the $n \times n$ nonsingular matrix A. The Frobenius norm of A is given by

$$\|A\|_F = \left(\sum a_{ij}^2\right)^{1/2}$$

(a) Construct the perturbation ∂A with smallest Frobenius norm such that $A - \partial A$ is singular. Hint: use one of the matrix decompositions of A.

learly

$$A - \partial A \quad U \; \Sigma - \quad V$$

is sin lar.

eno e e ol ns U $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$ and V $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$. e a a U and V are ni ary lies a $\underline{u}$ $\underline{v}$ 1 or j $1, \ldots, n$. e an ri e

$$\partial A \quad \sigma_n \underline{u}_n \underline{v}_n$$

and e Fro eni s no is

$$\partial A \;\; ^2 \quad \sigma_n^2 \sum \sum \quad \underline{u}_n \quad ^2 \; \underline{v}_n \quad ^2 \quad \sigma_n^2,$$

or

$$\partial A \quad \sigma_n$$

ose ∂A is any er a ion s a $A - \partial A$ is sin lar. en ere e is s a ve or ni len deno ed y $\underline{w}$ s a

$$A\underline{w} \quad \partial A\underline{w}.$$

o

$$\text{in} \frac{A\underline{z}}{\underline{z}} \quad \text{in} \;\; Aw \quad \sigma_n$$

s e lar es sin lar val e ∂A s e rea er an or e al o σ_n. in e li a ion y a ni ary a ri does no an e e Fro eni s no e Fro eni s no a eneral a ri is

$$A \quad \sum \sigma^2 \;\; ^1 \;\; ^2.$$

s

$$\partial A \quad \sigma_n.$$

d e ans er de ends on A. e alles sin lar val e A is ni e en e alles er a ion is ni e. ny o er er a ion $\partial \widehat{A}$ or i $A - \partial \widehat{A}$ is sin lar ill i sel ave a se ond non ero sin lar val e and s a lar er Fro eni s no . ere are li lies e alles sin lar val es A en ere are i le oi es ∂A i Fro eni s no e al o σ_n.

Numerical ODE:

5. Consider using forward Euler (same as AB1; Adams-Bashforth of first order) as a predictor, and the trapezoidal rule (same as AM2; Adams Moulton of second order) as a corrector for solving the ODE $y' = f(t, y)$.

   a. Write down the explicit steps that need to be taken in order to advance the numerical solution from time t_n to time $t_{n+1} = t_n + k$.

   b. Determine the order of the combined scheme. In case you know a theorem that gives the order directly, you may quote this *in its general form*, i.e. do not just state the answer in the present special case.

   c. The figure to the right illustrates the stability domain of the scheme. Prove that $(-2, 0)$ is the leftmost point

6. Partial Di erential Equations

onsider e s eady s a e adve ion di sion e a ion in one s a e di ension

$$-\partial_x\, a\, x\, \partial_x u\, x \quad b\, x\, \partial_x u \quad f, \qquad x \in 0,1$$

i o ndary ondi ions $u\ 0$ $u\ 1$ 0 and e ass ion a $a\ x$ is on in o s and $a\ x > 0$ or $x \in 0,1$

a es ri e e ni e di eren e F e od or a ro a in e sol ion sin en ered i eren es ind i eren es on e adve ion er . e h re resen e es s a in and ass e a ni or es . n ea ase a ove des ri e [redacted] e linear sys e s A [redacted] and A a e F e od yields.

ss e $a > 0$ and $b > 0$ are ons an . a e a rela ions i een a b and h a ass res e ei enval es o e linear sys are real or en ered [redacted] i eren es A and [redacted] ind i eren es A .

For ons an $a > 0$ $b > 0$ se Gers orin o nds o es a lis o nds on e ei enval es [redacted] o A e **upwind** di eren e a ri .

No onsider e ara oli e a ion ass e $a > 0$ and $b > 0$ are ons an

$$\partial\, u \quad a\partial_{xx} u\ x\ - b\partial_x u, \qquad x \in 0,1$$

d ri e e **Forward** ler s e or is e a ion sin en ered i eren es ind i eren es or e adve ion er .

e ri e a s l16en ered 21/F21 1.11511241 d00 /F11 /F21 1.2013110 d /F11 0.416310 ere ξ, η i x_{-1}, x_{+1} .

e en ered i eren e s en il or e se ond er is

$$b\ x \ \frac{-u\ x_{-1} \quad u\ x_{+1}}{2h} \quad b\ x\ \ u\ \ x \quad \frac{h^2}{12} b\ x\ \ u^{(3)}\ \eta \ ,$$

ere η i x_{-1}, x_{+1} .

e ind i eren e s en il or e se ond er is or $b\ x \ > 0$

$$b\ x \ \frac{-u\ x_{-1} \quad u\ x}{h} \quad b\ x\ \ u\ \ x \ - \frac{h}{2} b\ x\ \ u\ \ \eta \ ,$$

ere η i x_{-1}, x and or $b\ x \ < 0$

$$b\ x \ \frac{-u\ x \quad u\ x_{+1}}{h} \quad b\ x\ \ u\ \ x \quad \frac{h}{2} b\ x\ \ u\ \ \eta \ ,$$

ere η i x, x_{+1} .

i en ered di eren es e linear sys e is ridia onal deno ed y

$$A \quad \frac{1}{h^2} tri \left[-\ a\ x\ - h/2 \quad \frac{1}{2} b\ x \qquad a\ x\ - h/2 \quad a\ x \quad h/2 \qquad -\ a\ x \quad h/2\ - \frac{h}{2} b\ x \right]$$

For ind di eren es and ons an oe ien s $a >$