

# APPM GRADUATE PRELIMINARY EXAMINATION PARTIAL DIFFERENTIAL EQUATIONS – SOLUTIONS

Thursday August 24, 2017, 10AM –1PM

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There are five problems. Solve any four of the five problems. Each problem is worth 25 points.

On the front of your bluebook please write: (1) your name and (2) a grading table. Please start each problem with a new page. Text books, notes, calculators are NOT permitted. A sheet of convenient formulae is provided.

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## 1. (First order equations)

(a) (18 points)

Solve the first-order initial value problem

$$e^x \frac{\partial u}{\partial x} + ($$

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(b) When  $\alpha = 0$ , the PDE reduces to the ODE

$$(t + 1) \frac{du}{dt} = -u,$$

its general solution is

$$u = u_0(x) (t + 1)^{-1}$$

with  $u_0(x)$  independent of  $t$



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- (b) As  $t \rightarrow \infty$ , the solution  $u(x, t) \rightarrow 0$ . Since  $\phi(x)$  is nonzero only on a finite interval of  $x$ , the approximate solution for large  $t$  can be written as

$$u(x, t) \sim \frac{e^{-k_1^2 t}}{x + x_0} b_1 e^{-k_1^2 t} \sin \frac{x}{L},$$

i.e. it is determined by the lowest mode  $k_1 = \pi/L$ . The characteristic time of convergence to zero is  $\sim L^2/\pi^2$  and the time  $T$  is determined by  $u(L/2, t + T) = u(L/2, t)/2$ , i.e.

$$T = \frac{L^2}{\pi^2} \ln 2.$$

### 3. (Fourier series)

- (a) (10 pts)

Show that the pointwise convergent series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{1/2}}$$

cannot converge uniformly to a square integrable function  $f$  in  $[-\pi, \pi]$ .

- (b) (15 pts)

Let  $f(x)$  be  $2\pi$  periodic and piecewise smooth. Prove that its Fourier series converges uniformly and absolutely to  $f$ .

**Solution:**

- (a) Suppose the series converged uniformly to a square integrable function  $f$ . The Fourier coefficients of  $f$  are

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

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## 4. (Wave type equations)

Consider

$$\begin{aligned} u_{tt} - c^2 u_{xx} + au_t + \frac{a^2}{4}u &= 0, & 0 < x < L, & t > 0, \\ u(x, 0) = f(x), & u_t(x, 0) = g(x), & u(0, t) = u(L, t) &= 0, \end{aligned} \quad (4)$$

where  $f(x), g(x)$  are integrable and  $c > 0$  and  $a > 0$  are constants.

(a) (15 points)

Solve the above initial boundary value problem.

**Hint:** Look for solutions of the form  $u(x, t) = e^{-\frac{a}{2}t}w(x, t)$ .

(b) (5 points)

Derive the energy relation

$$\begin{aligned} \frac{dE}{dt} &= -2a \int_0^L u_t^2 dx, \\ E(t) &= \int_0^L u_t^2 + u_x^2 + \frac{a^2}{4}u^2 dx. \end{aligned} \quad (5)$$

What physical effect do the additional terms  $au_t$  and  $\frac{a^2}{4}u$  in (4) represent?

(c) (5 points)

Using energy relation (5), prove that the solution found in part (a) satisfies the energy relation (5).

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The boundary conditions  $u(0, t) = u(L, t) = 0$  imply  $u_t(0, t) = u_t(L, t) = 0$ . Performing integration-by-parts on the second term and applying these boundary conditions yields the desired energy relation

$$\frac{1}{2} \frac{d}{dt} \int_0^L (u_t^2 + u_x^2 + \frac{a^2}{4} u^2) dx = -a \int_0^L u_t^2 dx.$$

The energy  $E(t)$  is non-increasing in time, i.e.  $E(t_2) \leq E(t_1)$  for  $t_2 > t_1$ , indicating some dissipative force (e.g. friction, vibration) is modeled by the terms  $au_t$  and  $a^2u^2$ .

(c)

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