PDE Preliminary Examination: Spring 2012

Name: ______________________________

There are 5 problems. Each problem is worth 25 points. You are required to do 4 of them. Please indicate which 4 you choose. Note: Only 4 problems will be graded. A sheet of convenient formula is provided.

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

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1. Heat Equation

Assume that $u(x,t) \in C(\overline{Q}) \cap C^2(Q)$ $(Q = \{(x,t) \mid 0 < x < 1,\ t > 0\})$ is a solution to:

$$\begin{cases} u_t(x,t) = a u_{xx}(x,t) - F(x,t); & 0 < x < 1,\ t > 0,\ a > 0 \\ u(x,0) = f(x); & 0 < x < 1; \\ u(0,t) = 0; & t \geq 0; \\ u(L,t) = 0; & t \geq 0. \end{cases} \tag{1}$$

(a) State and prove a version of the maximum principle.

(b) Assume the solution is given by $u(x,t) = \int_0^1 g(x,y,t) f(y)\,dy$. In the case that $F(x,t) \equiv 0$, show that $g(x,y,t) = \int_0^1 g(x,z,t-s) g(z,y,s)\,dz$ for $t > s > 0$.

(c) State and prove a version of the uniqueness of solutions to (1).

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2. Fourier Series.

(a) Show explicitly a Fourier series and an open interval $S = (a,b)$ such that the series converges pointwise in S but does not converge uniformly in S.

(b) State the Weirstrass approximation theorem with any assumptions necessary.

(c) Suppose $f(x)$ is a continuous 2π periodic function. Prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(2\pi n \alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx \tag{2}$$

for any irrational α.

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3. Method of Characteristics.

Solve $(t^2 - 1)\, u_t(t,x) - x u_x(t,x) = u$, with the initial condition $u(0,x) = e^x$. (Solve all problems in terms of the original VARIABLES!)

TURN OVER

4. Wave equation.

Consider the forced wave equation

$$u_{tt} = u_{xx} \quad e^{\ x}; \qquad t > 0; \qquad 0 \quad x \quad L: \tag{3}$$

with initial conditions $u(x;0) = g(x); u_t(x;0) = 0$; and boundary conditions $u(0;t) = u(L;t) = 0$.

(a) Find a formal solution in terms of the function g.

(b) Find conditions on g that guarantee that the expression you found in (a) is a solut ion of the system.

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5. Laplace's Equation

Let $B = B_a(0) \quad R^2; a > 0$. Consider the f2.531.8Td27.5024(e)3.56148(r)341.911(t)0.649399(h)1.95024(e)327.689(f)1.94207(2.1671(u