A draft of INRIA lectures, May 1991

Wavelets, Multiresolution Analysis and Fast Numerical Algorithms

Bey n[1]

n od e on

e ec e e de o ed o f n e c o nd e e on o n e of po n e on c An y e ep e en on of de c of ope o n e e fo e p e C de on Zy nd o p do d . en ope o y e e ed e od fo e "co p e on e con e on o p fo e p y of e ep e en on con eq ence of e oc z on of e e n o p ce nd e n e do n n dd on e e on c e of e e e e p n on n o n e c en o n z on of n fo on on en e nd n e c on e een d . en ne o n e c n o n z on of o ne nd non ne n fo on een po e f oo on c An y y efe ed o L e ood ey nd C de on Zy nd eo e ee e . , nd no ppe o e n eq y po e f oo n N e c An y

n pp c on fo e p e n e p oce n . , nd n c . , e e on e od e e de e oped n e c fo e fo e n p oce n o ed on e . o e n fo A ec n q e of nd cod n e e c q d e o e M n od ced n . [7], c e e n oc on e o d o o e p e n y nd e f o ed e y of e o ono of f nc on p ce n f c e e e e e q c on on y o o c o ce e . o e nd e o e o e n pode n e of e p ce e n e o e f eq ency oc z on e efo e e e d o e y e de e . o e e e n n n e n e of o o ono con o e o c z on n e p ce e n e do n . e e. n e c nd o pp ed e d c n ed n e de e op en of o ono of e e . , . . , nd e no on of M e on An y . , . . [7], e e e ny ne con c on of o ono con o e oc z on n e p ce e n e do n no y . , . , . , . . [7],

n N e c An y ny n ed en of C de on Zy nd eo y e e ed n e . M po e o fo co p n po en n e c on . , . e . , e

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M po e o eq e o de N ope on o co p e e

$$p_j = -\sum_{i \neq j} \frac{q_i q_j}{| \; i}$$

e e od y e e ed de ce fo ed c n p d en eq on o
p e ne y e fo e co of n n e en y cond on n e of e e n
ce f n e d of n e d ence o n e e e en ep e en on e e e ep
e en on of e de e n e e en p e od c on

[illegible]

II.1 Multiresolution analysis.

e de n on of e e o on n y no on n od ced y Meye , nd M , c p e e e en

f ee n en e of e en o o of ene y e
o e ene e n e d of e e en e

$$\mathbf{V_n} \subset \quad \subset \mathbf{V} \subset \mathbf{V} \subset \mathbf{V} \quad \mathbf{V} \subset \mathbf{L} \quad \mathbf{R^d}$$

n n e c e z on e p ce **V** n e d en on

II.2 The Haar basis

o on e e on y e p e of e e o on n y y n De n on Cond on e Co p n n o p e of e o e de e n e ec e nd p o de ef p o o ype fo n e c e pe en on f d en j;k −j= −j — ∈ **Z** fo ed y e d on nd n on of n e f nc on

$$\begin{cases} & \text{fo} \\ - & \text{fo} \quad \leq \\ & \text{e e e} \end{cases}$$

n c e e e e c c e c f nc on of e n e o e c j;k −j= −j — ∈ **Z** e of $\mathbf{V_j}$ nd j;k −j= −j — ∈ **Z** e of $\mathbf{W_j}$

e deco po on of f nc on n o n o de N p oced e en N n pe of f nc on c y fo p c y e o of e of ed e e of f on n e of en −n

$$_k = {}^{n=} \int_{-n\,k}^{-n\,k+)} f \; d$$

e o n coe c en

$$d_k^{j+} = \frac{}{\sqrt{}} \; {}^{j}_{k-} - {}^{j}_{k}$$

nd e e

$$_k^{j+} = \frac{}{\sqrt{}} \; {}^{j}_{k-} \quad {}^{j}_{k}$$

fo $n -$ nd n−j− — e y o ee e n e o e of coe c en d_k^j $_k^j$ n eq e $N-$ dd on nd N p c on n o d en on e e e o n o e nd d fo e econd de ned y e of ee nd of f nc on ppo ed on q e $_{j;k}$ $_{j;k'}$ y $_{j;k}$ $_{j;k'}$ y nd $_{j;k}$ $_{j;k'}$ y e e e c c e c f nc on of e n e nd $_{j;k}$ $:=$ $^{-j=}$ $^{-j}$ $-$ ep e en n n ope o n e d o e non nd d fo e e no o y eco e c e e

By con de n n n e ope o

$$f \quad := \int \quad y \; f \; y \; dy$$

nd e p nd n e ne n o d en on e nd fo C de on Zy nd nd p e do d en ope o e dec y of en e f nc on of e d nce f o e d on f e n e e ep e en on n n e o n e ne e e c of ope o e en y n e o d on e ne e oo y f o e d on o e p e e ne y of C de on Zy nd ope o fy e e e

$$| \quad y | \leq \frac{}{| \; - y|}$$

$$| \partial_x^M \quad y | \quad | \partial_y^M \quad y | \leq \frac{C_M}{| \; - y|^{\; +M}}$$

fo o e $M \geq$ Le $M :=$ n nd con de

$$^{j}_{kk'} := \int\int \quad y \quad_{j;k} \quad_{j;k'} \; y \; d \; dy$$

e e e e e d nce e een $| \; - \; '| \geq$ nce

$$\int \quad_{j;k} \quad d \; := \qquad\qquad e$$

e e

$$| \quad ; \quad y \quad f \; r \quad x$$

e on e n n e dec y n c en o e co p n n p c c
o e f e dec y nece y o e f nc on e e n n
o en e n n o en e e pon e fo n n p c c o e
con o n e con n n e co p e y e e of e f o

II.3 Orthonormal bases of compactly supported wavelets

e q e on of e e ence of e on n y oo f nc on of
Cond on e no e ed n e con c on of e o ono e e
ene z n f nc on y o e nd Meye e con de
on y co p c y ppo ed e e n n o en con c ed y D ec e
fo o n e o of Y Meye nd M o o of e e
e n d fo o e c o ce of e e e e f e e c o
c o ce of e e y on y con de n fo d en on $d \geq$ e con c ed
f o o e fo d —

Le con de e e on n y fo $L^2(\mathbb{R}^7)$ n f d d e 7 econd e o ol on y of $\{ \quad - \quad \}_{k \in \mathbb{Z}}$ p e

$$k = \int_{-\infty}^{+\infty} \quad - \quad d = \int_{-\infty}^{+\infty} | \quad | \, e^{-ik} \, d$$

nd e efo e

$$k = \int \sum_{l \in \mathbb{Z}} | \quad \quad | \, e^{-ik} \, d$$

nd

$$\sum_{l \in \mathbb{Z}} | \quad \quad | =$$

nl e o n

$$\sum_{l \in \mathbb{Z}}$$

no d c o o ee e . , . , . . , on e c ed e ∈ **Z** e
e e { $_{j;k}$ — $^{-j=}$ $^{-j}$ — $\}_{k \in \mathbf{Z}}$ fo n o ono of $\mathbf{W_j}$
e fo o n e D ec e , c c e ze ono e c po yno
on of c co e pond o e o ono e of co p c y ppo ed
e e n n o en

Lemma II.1 Any trigonometric polynomial solution of (2.26) is of the form

$$-\tfrac{1}{2} \quad e^{i} \quad {}^{M} \quad e^{i}$$

where $M \geq$ **is the number of vanishing moments, and where is a polynomial, such that**

$$| \quad e^{i} \quad | - P \quad n \quad \tfrac{1}{2} \quad n^{M} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \cos$$

where

$$P \; y \; - \sum_{k}^{M-} \; M - \quad y^{k}$$

and is an odd polynomial, such that

$$\leq P \; y \quad y^{M} \quad \tfrac{1}{2} - \quad d \qquad\qquad f$$

e e $^{j}_{k}$ nd d^{j}_{k} y e e ed pe od c eq ence e pe od $n-j$ Co p n

nd ed y e py d e e e

$$\{\ _{k}\} \longrightarrow \{\ _{k}\} \longrightarrow \{\ _{k}\} \longrightarrow \{\ _{k}\} \cdots$$

$$\searrow \qquad \searrow \qquad \searrow$$

$$\{d_{k}\} \qquad \{d$$

e en de ne $f_{\mathbf{m}}$:= $f_{\mathbf{m}}$ − $\mathbf{m}$ · f e e $\mathbf{m}$ c o en o $\langle f_{\mathbf{m}}$ $\mathbf{M} \rangle$:= fo

:= M c e n e de ed o o on y o $\mathbf{M}$ y e con n e o e e $\mathbf{V}_j^{M}$ en n e nd e p ce $\mathbf{W}_j^{M;}$ e o on co p e en of $\mathbf{V}_j^{M;}$ n $\mathbf{V}_{j-}^{M;}$

$$\mathbf{V}_{j-}^{M;} = \mathbf{V}_j^{M;} \oplus \mathbf{W}_j^{M;}$$

e p ce $\mathbf{W}^{M;}$ p nned y e o ono

$$\{ \quad i \quad l\, y \quad i \quad l\, y \quad i \quad l\, y \quad = \quad M\}$$

e e $\{\ {}_m\}_m^{m\ M}$ n o ono fo $\mathbf{V}^{M}$ c e e en on e M e e en n n o en e o on o e po yno $\ {}^i y^l \ = \ M -$

e p ce $\mathbf{W}_j^{M;}$ p nned y d on nd n on of e f nc on of $\mathbf{W}^{M;}$ nd e of $\mathbf{L}$, con of e f nc on nd e o o de po yno $\ {}^i y^l \ = \ M -$

e no e e o d en on e e eq e M d en co n on of one d en on f nc on e e M e n e of n n o en On e o e nd e o d en on o ned y n co p c y ppo ed e e , eq e on y ee c co n on c p e e con c on of e non nd d fo ee ec on

II.5 A remark on computing in the wavelet bases

n y e no e once e e een c o en co p e e y de e ne e f nc on nd nd e efo e e e on n y n n e e n o e on n p ope y con c ed o e f nc on nd e ne e co p ed D e o e ec e de n on of e e e e n p on e pe fo ed e q d e o e nd e en f ey n o e q n e c ed nd A n e p e e co p e e o en of e n f nc on e e p e on fo e o en

$$\mathcal{M}_\infty^{m} = \int {}^{m} \quad d \quad = \quad M -$$

n e of e e coe c en $\{\ {}_k\}_k^{k\ L}$ y e fo nd n fo fo

$$= \quad - = \quad \quad -j$$

e e

$$= \quad - = \sum_k {}^{k}\ {}_k e^{ik}$$

e o en $\mathcal{M}_{\infty}^{m}$ e o ned n e c y n e de ed cc cy y ec e y ene n eq ence of ec o $\{\mathcal{M}_{r}^{m}\}_{m}^{m}$ M − fo r −

$$\mathcal{M}_{r+}^{m} - \sum_{j}^{m} \quad ! \quad -jr\,\mathcal{M}_{r}^{m-j}\mathcal{M}^{j}$$

n

$$\mathcal{M}^{m} - \;{}^{-m-\frac{1}{2}} \sum_{k}^{k\;k-} \quad k \quad m \quad - \qquad M -$$

c ec o $\{\mathcal{M}_{r}^{m}\}_{m}^{m}$ M − ep e en M o en of e p od c n r e nd e on con e e p d y No ce e ne e co p ed e f nc on e f

e non nd d nd nd d fo

III.1 The Non-Standard Form

Le e n ope o

$$\mathbf{L} \quad \mathbf{R} \rightarrow \mathbf{L} \quad \mathbf{R}$$

e e ne y De n n p o ec on ope o on e p ce $\mathbf{V_j}$ $\in \mathbf{Z}$

$$P_{\mathbf{j}} \quad \mathbf{L} \quad \mathbf{R} \rightarrow \mathbf{V_j}$$

$$P_{\mathbf{j}} f \quad = \sum_{\mathbf{k}} \langle f \quad {}_{\mathbf{j;k}} \rangle \quad {}_{\mathbf{j;k}}$$

nd e p nd n n e e op c e e e o n

$$= \sum_{\mathbf{j} \in \mathbf{Z}} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j}$$

e e

$$\mathbf{j} = P_{\mathbf{j}-} - P_{\mathbf{j}}$$

e p o ec on ope o on e p ce $\mathbf{W_j}$ f e e e co e n en n e d of e e e

$$= \sum_{\mathbf{j} \quad -\infty} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j} \quad P_{\mathbf{n}} \quad P_{\mathbf{n}}$$

nd f e e = e ne e en

$$= \sum_{\mathbf{j}} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j} \quad P_{\mathbf{n}} \quad P_{\mathbf{n}} \qquad 7$$

e e $\sim$ $= P \quad P$ d e z on of e ope o on e ne e p n on e nd 7 deco po e ope o n o of con on f o d en e e

e non nd d fo ep e en on e 7 of e ope o c n of p e

$$= \{A_{\mathbf{j}} \quad B_{\mathbf{j}} \quad {}_{\mathbf{j}}\}_{\mathbf{j} \in \mathbf{Z}}$$

c n on e p ce $\mathbf{V_j}$ nd $\mathbf{W_j}$

$$A_{\mathbf{j}} \quad \mathbf{W_j} \rightarrow \mathbf{W_j}$$

$$B_{\mathbf{j}} \quad \mathbf{V_j} \rightarrow \mathbf{W_j}$$

$\mathbf{W_j} \to \mathbf{V_j}$

e e e ope o $\{A_j\ B_j\ \ j\}_{j\in Z}$ e de ned A_j := j j B_j := j P_j nd j := $-P_j$ j

e ope o $\{A_j\ B_j\ \ j\}_{j\in Z}$ d ec e de n on e e on

$$j := \begin{pmatrix} A_{j+} & B_{j+} \\ j+ & j+ \end{pmatrix}$$

e e ope o j := $-P_j$ P_j

$$j \quad \mathbf{V_j} \to \mathbf{V_j}$$

nd e ope o ep e en ed y e × n pp n

$$\begin{pmatrix} A_{j+} & B_{j+} \\ j+ & j+ \end{pmatrix} \quad \mathbf{W_{j+}} \oplus \mathbf{V_{j+}} \to \mathbf{W_{j+}} \oplus \mathbf{V_{j+}}$$ e

f e e co e n en

$$:= \{\{A_j\ B_j\ \ j\}_{j\in Z\ j\le n}\ \ n\}$$

e e n := $-P_n$ P_n f e n e of e n e en := n n nd e ope o e o nzed oc of e ee e nd

Le e e fo o n o e on

e ope o A_j de e e n e c on on e e ony nce e p ce $\mathbf{W_j}$ n n e e en of e d ec n

e ope o B_j j n nd de e e n e c on e een e e nd co e e ndeed e p ce $\mathbf{V_j}$ con n e p ce $\mathbf{V_{j'}}$ ' ee

e ope o j n e ed e on of e ope o j−

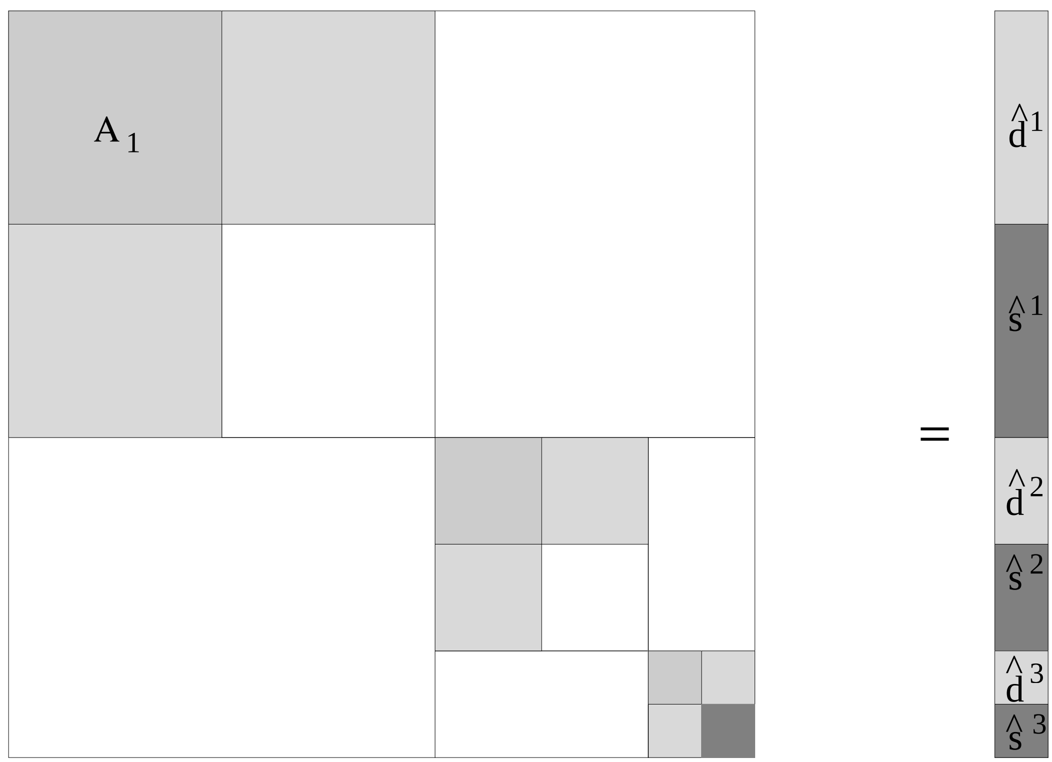
e ope o A_j B_j nd j e ep e en ed y e ce j j nd j

$$^{j}_{k;k'} := \int\int \quad y \quad j;k \quad j;k' \ y \ d \ dy$$

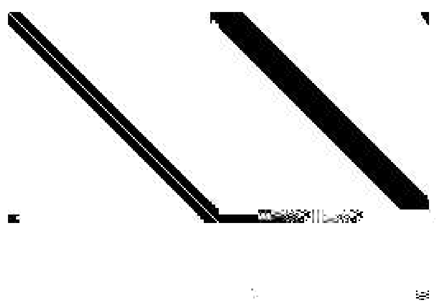
$$^{j}_{k;k'} := \int\int \quad y \quad j;k \quad j;k' \ y \ d \ dy \qquad 7$$

nd

$$^{j}_{k;k'} := \int\int \quad y \quad j;k \quad j;k' \ y \ d \ dy$$



A 1
=
d^1
s^1
d^2
s^2
d^3
s^3



e An e p e of n e non nd d fo ee p e

e ope o j ep e en ed y e j

$$\mathbf{Z}\ \mathbf{Z}$$

$$^{\mathbf{j}}_{\mathbf{k;k'}} - \quad y \quad _{\mathbf{j;k}} \quad _{\mathbf{j;k'}}\ y \quad d \quad dy$$

en e of coe c en $_{\mathbf{k;k'}}$ $' -$ $N -$ epe ed pp c on of e [redacted]
fo 7 p od ce

[redacted]

$$^{\mathbf{j}}_{\mathbf{i;l}} - \sum^{\mathbf{L}}_{\mathbf{k;m}} \quad \mathbf{k} \quad \mathbf{m} \quad ^{\mathbf{j}-}_{\mathbf{k}+} \quad \mathbf{i}+ \quad \mathbf{;m}+ \quad \mathbf{l}+$$

III.2 The Standard Form

e nd d fo o ned y ep e en n

$$\mathbf{V}_{\mathbf{j}} = \bigoplus_{\mathbf{j}'>\mathbf{j}}^{\mathsf{M}} \mathbf{W}_{\mathbf{j}'}$$

nd con de n fo e c e e ope o $\{B_{\mathbf{j}}^{\mathbf{j}'}, \Gamma_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}'>\mathbf{j}}$

$$B_{\mathbf{j}}^{\mathbf{j}'} \quad \mathbf{W}_{\mathbf{j}'} \to \mathbf{W}_{\mathbf{j}}$$

$$\Gamma_{\mathbf{j}}^{\mathbf{j}'} \quad \mathbf{W}_{\mathbf{j}} \to \mathbf{W}_{\mathbf{j}'}$$

f e e e co e en n en n e d of e e

$$\mathbf{V}_{\mathbf{j}} = \mathbf{V}_{\mathbf{n}} \bigoplus_{\mathbf{j}'\ \mathbf{j}+}^{\mathbf{j}'\mathsf{M}\mathbf{n}} \mathbf{W}_{\mathbf{j}'}$$

n c e e ope o $\{B_{\mathbf{j}}^{\mathbf{j}'}, \Gamma_{\mathbf{j}}^{\mathbf{j}'}\}$ fo $j' =$ n e e e n nd nd n dd on fo e c e e e ope o $\{B_{\mathbf{j}}^{\mathbf{n}+}\}$ nd $\{\Gamma_{\mathbf{j}}^{\mathbf{n}+}\}$

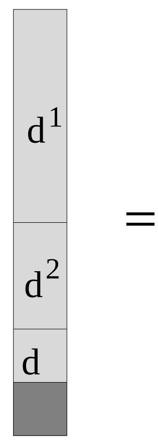
$$B_{\mathbf{j}}^{\mathbf{n}+} \quad \mathbf{V}_{\mathbf{n}} \to \mathbf{W}_{\mathbf{j}}$$

$$\Gamma_{\mathbf{j}}^{\mathbf{n}+} \quad \mathbf{W}_{\mathbf{j}} \to \mathbf{V}_{\mathbf{n}}$$

n no on $\Gamma_{\mathbf{n}}^{\mathbf{n}+} = \Gamma_{\mathbf{n}}$ nd $B_{\mathbf{n}}^{\mathbf{n}+} = B_{\mathbf{n}}$ f e n e of e n e nd $\mathbf{V}$ n e d en on en e nd d fo ep e en on of $= P$ P

$$= \{A_{\mathbf{j}}\ \{B_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}'\ \mathbf{j}+}^{\mathbf{j}'\ \mathbf{n}}\ \{\Gamma_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}'\ \mathbf{j}+}^{\mathbf{j}'\ \mathbf{n}}\ B_{\mathbf{j}}^{\mathbf{n}+}\ \Gamma_{\mathbf{j}}^{\mathbf{n}+}\ \mathbf{n}\}_{\mathbf{j}\ ;:::;\mathbf{n}}$$

e ope o e o n zed oc of e ee e nd e e f e ope o C de on Zy nd o p e do o



d^1
d^2
d
=

e co p e on of ope o

e co p e on of ope o o n o e o d e con c on of e p e ep e
en on n o ono e d ec y . ec e peed of co p on o
e e co p e on of d of e fo e p e c e ed y e od o e n
nd n p e ep e en on n e y e deq e fo e pp c on e
co p e on of ope o c fo ep e en on n n o de o e. ec e y co
p e n e p e fo e nd d nd non nd d fo of ope o n e e e
e y e e ed co p e on e e fo de c of nd non nd d fo of ope

the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| \, ^j_{i;l}| \quad | \, ^j_{i;l}| \quad | \, ^j_{i;l}| \leq \frac{C_M}{| \; - \; |^{M+}}$$

for all $| \; - \; | \geq \; M$.

con de on pp y n e c e of p e do d en ope o Le e p e do d en ope o y o de ned y e fo

$$f \quad - \; e^{ix} \qquad f \quad d \; - \qquad y \; f \; y \; dy$$

e e e d on e ne of

Proposition IV.2 **If the wavelet basis has M vanishing moments, then for any pseudo-di erential operator with symbol of and * of * satisfying the standard conditions**

$$| \quad _x \quad | \leq C \; ; \qquad | \quad - \; +$$

$$| \quad _x \; ^* \quad | \leq C \; ; \qquad | \quad - \; +$$

the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| \, ^j_{i;l}| \quad | \, ^j_{i;l}| \quad | \, ^j_{i;l}| \leq \frac{^j \; C_M}{| \; - \; | \; ^{M+}}$$

for all integer , .

f e pp o e e ope o ^{N} y e ope o $^{N;B}$ o ned f o ^{N} y e n o ze o coe c en of ce $^j_{i;l}$ $^j_{i;l}$ nd $^j_{i;l}$ o de of nd of d $B \geq \; M$ o nd e d on en e y o ee

$$|| \; ^{N;B} - \; ^N || \leq \frac{C}{B^M} \; \text{o} \; N$$

e e C con n de e ned y e ne n o n e c pp c on e cc cy of c c on ed nd e p e e of e o n o c e e nd d B nd o de M e o e c o en n c nne e de ed p ec on of c c on c e ed f M ed en B o e c

$$|| \; ^{N;B} - \; ^N || \leq \frac{C}{B^M} \; \text{o} \; N \leq$$

y $-$ * y e

e e f c y n y n e f nc on e e nd e nd e efo e e ope o L nd L^* po e o dec de f C de on Zy nd ope o o nded

Theorem IV.1 (G. David, J.L. Journe) **Suppose that the operator (3.1) satis es the conditions (4.5), (4.6), and (4.16). Then a necessary and su cient condition for to be bounded on L is that in (4.24) and y in (4.25) belong to dyadic BMO, i.e. satisfy condition**

$$\sup_J \frac{1}{|J|} \int_J | \quad J \quad | \, d \leq C \qquad e$$

where is a dyadic interval and

$$J \quad - \frac{}{| \; |} \int_J \quad d \qquad e \; 7$$

p n e ope o n o e of ee e e nd e n e p e y e d o e e e e e no e e f nc on nd * e e y co p ed n e p oce of con c n e non nd d fo nd nd y e ed o p o de f e e of e no of e ope o

e d e en ope o n e e e

V.1 The operator d=dx in wavelet bases

e non nd d fo of e e co on y ed ope o y e co p ed e p c y e con c e non nd d fo of e ope o d d e e e en of A_j B_j nd r of e e d d d

e e n e e oco e on coe c en of e e $\{ k \}_k^k$ $L-$

$$n \quad \sum_{i}^{L-n} \quad i \quad i+n \quad n \quad L-$$

e y o ee e oco e on coe c en n e en nd ce e ze o

$$k \qquad L -$$

y e e fyed y n o co p e nd |

$$| \quad - \quad \sum_{n}^{L} \quad n \ \text{co} \ n$$

$$| \quad - - - \sum_{k}^{L=} \quad k- \ \text{co} \quad - \quad - \sum_{k}^{L=} \quad k \ \text{co}$$

e e n e en n Co n n nd o fy e o n

$$\sum_{k}^{L=} \quad k \ \text{co} \quad -$$

nd ence nd

e e en o en of e coe c en k- f o n n e y

$$\sum_{k}^{k \ L=} \quad k- \quad - \quad m \quad \text{fo} \quad \leq \quad \leq M - \qquad e$$

nce

$$- \quad m \quad | \quad \text{fo} \quad \leq \quad \leq M -$$

c fo o f o e Co o d en f e e d e y fy o 7 7 e f d cof e n e e e 7

$$r_i = \sum_{k}^{L} \sum_{m}^{k-k+} {}_{k\,k-m} r_{i+m}$$

C n n e o de of on n nd n e f c $\sum_{k}^{L-}$ $k =$ e
e

$$r_l = r_l \sum_{n}^{L} {}_n r_{l-n} \quad r_{l+n} \quad \in \mathbf{Z}$$

e e $_n$ e en n n e o n e f o
n o de o o n e e fo o n e on

$$\sum_{l=-\infty}^{+\infty} {}_m \quad - \quad = {}_m \sum_{l}^{m} - {}_l \quad M_l^{\,} {}_{m-l} \;!$$

e e

$$M_l = \int_{-\infty}^{+\infty} {}^{l} d \quad =$$

e e o en of e f nc on e on fo o p y on n o e
n fo nd n Le n z e n nd = e o n
f $M \geq$ en

$$| \;| \;| \leq C \quad | \;|^{- -}$$

e e nd ence e n e n e y con e en e on
fo o f o Le of e e o n

$$| \;| \leq C \quad | \;|^{-M+\log_2 B}$$

e e

$$B = \sup_{\in \mathbf{R}} | \quad e^{i} \quad |$$

D e o e cond on e e o $B = M - \;-$ o e
e e ence of o on of e y e of eq on e nd fo o
f o e e ence of e n e n nce e n f nc on co p c
ppo e e e on y 7 7 d ee e d o e

∞∈{∈{∞⟩∞∞∈{∞∈∞∞|∞|∈

e e

$$r(\xi) := \sum_{l} r_l e^{il}$$

$$r_{even}(\xi) := \sum_{l} r_{l} e^{il} \qquad (7$$

nd

$$r_{odd}(\xi) := \sum_{l} r_{l+} e^{i(l+)/}$$

No c n

$$r_{even}(\xi) := r(\xi) \quad r(\xi)$$

nd

$$r_{odd}(\xi) := r(\xi) - r(\xi) \qquad (e$$

nd n e o n f o

$$r(\xi) := r(\xi) \quad r(\xi) \quad \left[\quad (\xi) \;|\; - \quad r(\xi) - r(\xi) \right] \qquad (e$$

n y n e e

$$r(\xi) := \quad (\xi) \;|\; r(\xi) \quad (\xi) \quad |\; r(\xi) \qquad (e$$

e n $(\xi) :=$ n e e o n r $:= r$ nd e n q ene of e o on of e nd fo o f o e n q ene of e ep e en on of d d en e o on r_l of e nd e con de e ope o $_j$ de ned y e coe c en on e p ce V_j nd pp y o c en y oo f nc on f nce $r^j_l := {}^{-j} r_l$ e e e

$$_j f := \sum_{k \in Z} \left({}^{-j} \sum_{l} r_l f_{j;k-l} \right) {}_{j;k} \qquad (e$$

e e

$$f_{j;k-l} := {}^{-j=} \int_{-\infty}^{+\infty} f \quad {}^{-j} \quad - \quad d \qquad (ee$$

e n ee

$$f_{j;k-l} :=$$

d 7 7

e e — j nd | − | ≤ j n e n e nd n e nd e o n

$$_j f = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{+\infty} f' \quad {}_{j;k} \; d \quad {}_{j;k}$$

$$\sum_{k\in\mathbb{Z}} j \sum_{l} r_l \int_{-\infty}^{+\infty} f'' \quad {}_{j;k} \; d \quad {}_{j;k} \quad !$$

c e → −∞ ope o j nd d d co nc de on oo f nc on n nd d en e y o p o e −∞ = d d nd ence e o on o e nd n q e e e on 7 fo o no f o

Remark 2 e no e e p e on nd fo l nd l l = − − l y e p ed y c n n e o de of on n nd nd n od c n e co e on coe c en $\sum_{i}^{L-n}$ i i+n $\sum_{i}^{L-n}$ i i+n nd $\sum_{i}^{L-n}$ i i+n e e p e on fo l e pec y p e l = e r l − r l

Examples. o e e p e e D ec e e e con c ed n e co p e e coe c en k = M e e M e n e of n n o en nd L = M n e on e of

$$= - \frac{M-}{M- \quad M-} \int n^{M-} d$$

e nd y co p n R n M− d

$$= - C_M \sum_{m}^{M} \frac{- \;^{m-} \; co \; -}{M \quad M \quad - \quad -}$$

e e

$$C_M = \left[\frac{M -}{M - \quad e^{M-}} \right]$$

y co p n e nd e e

$$_{m-} = \frac{- \;^{m-} C_M}{M \quad M \quad - \quad -} \qquad e e = M$$

e no e y e of e n o on of ne y e on co e c en m− n e on y con c on e coe c en r l e on n e e coe c en r l e e e fo D ec e e e ed n e of n n o en M e e e e e e e fo en

j e d e d en

j ene e of e p d o yno e

o n eq on of opo on e p e en e e fo D ec e e e

M e

1 M

nd

r r

e coe c en of e p e c n e fo nd n ny oo on n e c n y c o ce of coe c en fo n e c d en on

---

2 M

7

e

nd

7

r r r r_4

---

3 M e

e e e e e e e

nd

7

r e r e e r e e

r_4 e 7 r 7 e r 7

---

4 M

e 7 e 7 7

e

nd

7 7

r e 7 r e 7 r 7

7 7

r_4 e r

5 M =

and

r = ... r = ... r = ...

r_4 = ... r = ... r = ...

r = ... r = ... r = ...

r = ...

Coe c en s fo M = nd M = c n e co p ed e co e pond n o p f o e fo o n e e o

Iterative algorithm for computing the coe cients r_l.

A y of on eq on e nd e y o e n e e o e r = nd r = − nd e e n e o eco p e r_l e y o e fy n e nd e ed d e o e c o ce of n z on e fo o n fo D ec e e e M = co p ed n o d p y e coe c en $\{r_l\}$ e no e r_{-l} = $-r_l$ nd r =

V.2 The operators d^n/dx^n in the wavelet bases

o e ope o d d e non nd d fo of e ope o d^n d ^{n} co p e e y de e ned y ep e en on on e p ce $\mathbf{V}$ e y e coe c en

$$r_l^{(n)} = \int_{-\infty}^{+\infty} \ldots \frac{d^n}{d^n} \ldots d \qquad \in \mathbf{Z}$$

o e n e y

$$r_l^{(n)} = \int_{-\infty}^{+\infty} \xi^n |\ldots \xi| \, e^{-il} \, d\xi$$

f e ne n o e ee e e o

		Coe cients
	l	ı
M = 5	1	-0.82590601185015
	2	0.22882018706694
	3	-5.3352571932672E-

		Coe cients
	l	ı
M = 8	1	-0.88344604609097
	2	0.30325935147672

Proposition V.2 **1. If the integrals in (5.52) or (5.53) exist, then the coe cients $r_l^{(n)}$, $l \in \mathbf{Z}$ satisfy the following system of linear algebraic equations**

$$r_l^{(n)} = {}^{n} \left[r_{l} - \sum_{k}^{L} {}_{k-} \left(r^{(n)}_{l- k+} \quad r^{(n)}_{l+ k-} \right) \right]$$

and

$$\sum_{l} {}^{n} r_l^{(n)} = - {}^{n} n$$

where ${}_{k-}$ are given in (5.19).

2. Let $M \geq n$, where M is the number of vanishing moments in (2.16). If the integrals in (5.52) or (5.53) exist, then the equations (5.54) and (5.55) have a unique solution with a nite number of non-zero coe cients $r_l^{(n)}$, namely, $r_l^{(n)} \neq$ for $-L \leq l \leq L -$. Also, for even n

$$r_l^{(n)} = r_{-l}^{(n)}$$

$$\sum_{l} {}^{n} r_l^{(n)} = n = n -$$

and

$$\sum_{l} r_l^{(n)} =$$

and for odd n

$$r_l^{(n)} = -r_{-l}^{(n)}$$

$$\sum_{l} {}^{n-} r_l^{(n)} = n = \qquad \leq l \leq L$$

$$A \quad e\, M$$

e no e on e e e L = e e e e o n n
o en M = do no e e e [illegible] o de e ponen ee , e ep e en on
of e d de e e on y f e n e of n n o en M =

e eq on fo co p n e coe c en $r_l^{(n)}$ y e e ed n e en e
p o e Le de e e eq on co e pond n o e fo d^n d ^{n} d ec y f o
e e e

$$r_l^{(n)} = \int \sum_{k\in Z} | \; | ^n \; ^n e^{-il} \, d$$

e efo e

$$r = \sum_{k\in Z} | \; | ^n \; ^n$$

e e

$$r = \sum_l r_l^{(n)} e^{il}$$

n e e on

e

n o e nd de of nd n o e e en nd odd nd ce n
ep e y e e

$$r = ^n \; | \; r \; | \; r$$

Le con de e ope o M on pe od c f nc on d f n f d

$$M f = | \; f \; |$$

n ee e e n e c e dence n f c of n e e nce ep
e en one of e d n e of co p n n e ee e
e e c e

N	[illegible]	[illegible]p
64	0.14545E+04	0.10792E+02
128	0.58181E+04	0.11511E+02
256	0.23272E+05	0.12091E+02
512	0.93089E+05	

e con o on ope o n e e e

n ec on e con de e co p on of e non nd d fo of con o on ope o . o con o on ope e q d e fo fo ep e en n e e ne on **V** e of e p e fo d e

nd e den y fo o f o
o d

nce e o en of e f nc on n eq on e e d o
po n q d e fo fo co p n e ep e en on of con o on
e ne e fo co p c y ppo ed e e fo o
e e nne fo e pec c o ce of e e e de eqn
e e e f ed o en of e f nc on n e ef p pe fo
e de

e e e n od ce d en pp o c c con n of
ne e c eq on ec o y p o c cond on pe
c y p e f e y o of e ope o o o eneo of e
c e ope o co p e e y de ned y ep e en on on **V**
Le con de o e p e of c ope o e nd
ope o of f c on d en on o n d en on

VI.1 The Hilbert Transform

e pp y o e od o e co p on of e non nd d fo e n
fo

$$:= \mathcal{H}f \quad y := \text{p} \int_{-\infty}^{\infty} \frac{f}{-} d$$

e e p deno e p nc p e :=

e ep e en on of $\mathcal{H}$ on **V** de ned y e coe c en

$$r_{\mathbf{l}} := \int_{-\infty}^{\infty} - \quad \mathcal{H} \quad d \quad \in \mathbf{Z}$$

c n n co p e e y de ne o e coe c en of e non nd d

$\mathcal{H} := \{A_{\mathbf{j}} \; B_{\mathbf{j}} \; \mathbf{j}\}_{\mathbf{j} \in \mathbf{Z}}$ $A_{\mathbf{j}} := A$ $B_{\mathbf{j}} := B$ nd $\mathbf{j} :=$ e e e

	l	Coefficients r_l	l	Coefficients r_l
M = 6	1	-0.588303698	9	-0.035367761
	2	-0.077576414	10	-0.031830988
	3	-0.128743695	11	-0.028937262
	4	-0.075063628	12	-0.026525823
	5	-0.064168018	13	-0.024485376
	6	-0.053041366	14	-0.022736420
	7	-0.045470650	15	-0.021220659
	8	-0.039788641	16	-0.019894368

e e e coe c en r_l of e n fo fo D ec e e e n n o en

e coe c en $r_l \in \mathbf{Z}$ n fy e fo o n y e of ne e c eq on

$$r_l - r_l - \sum_{k} \; {}_{k-} \; r_{l-k+} \; r_{l+k-}$$

e e e coe c en ${}_{k-}$ e en n n e nd e o n e y p o c of r_l fo e

$$r_l - - \; O \; \frac{}{M}$$

By e n n e of

$$r_l - \int^{\infty} | \; | \; n \; d$$

e o n $r_l - r_{-l}$ nd r e no e e coe c en r c nno e de e ned f o eq on e nd o n e e y p o c cond on e co p e e coe c en $r_l \neq$ ny p e ed cc cy

Example.

e co p e e coe c en r_l of e n fo fo D ec e e e n n o en cc cy e coe c en fo e o ned n y p o c e no e $r_{-l} - r_l$ nd r

VI.2 The fractional derivatives

e e fo o n de n on of f c on de e

$$ {}_{x}f \;=\; \int_{-\infty}^{+\infty} \frac{- y \; - \; + \; -}{-} \; f \; y \; dy $$

e e e con de f en de ne f c on n de e e ep e en on of ${}_{x}$ on $\mathbf{V}$ de e ned y e coe c en

$$ r_{l} = \int_{-\infty}^{+\infty} - \quad {}_{x} \quad d \quad \in \mathbf{Z} $$

p o ded n e e

e non nd d fo ${}_{x} = \{A_{j} \; B_{j} \; {}_{j}\}_{j \in Z}$ co p ed $A_{j} = -^{j} A$ $B_{j} = -^{j} B$ nd ${}_{j} = -^{j}$ e e e e en ${}_{i-l}$ ${}_{i-l}$ nd ${}_{i-l}$ of A B nd e o ned f o e coe c en r_{l}

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} \quad {}_{k} \; {}_{k'} \; r_{\, i+k-k'} $$

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} \quad {}_{k} \; {}_{k'} \; r_{\, i+k-k'} $$

nd

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} \quad {}_{k} \; {}_{k'} \; r_{\, i+k-k'} $$

e y o e fy e coe c en r_{l} fy e fo o n y e of ne e c eq on

$$ r_{l} = \left[4 r_{\, l} \; - \sum_{k}^{L} \; {}_{k-} \; r_{\, l-\; k+} \; r_{\, l+\; k-} \right] $$

e e e coe c en ${}_{k-}$ e en n n e nd e o n e y p o c of r_{l} fo e

$$ r_{l} = \frac{}{-} \; \frac{}{+} \quad O \; \frac{}{+ \; + \; M} \quad \text{fo} $$

$$ r_{l} = \quad \text{fo} $$

Example.

	i	Coefficients	i	Coefficients
M = 6	-7	-2.82831017E-06	4	-2.77955293E-02
	-6	-1.68623867E-06	5	-2.61324170E-02
	-5	4.45847796E-04	6	-1.91718816E-02
	-4	-4.34633415E-03	7	-1.52272841E-02
	-3	2.28821728E-02	8	-1.24667403E-02
	-2	-8.49883759E-02	9	-1.04479500E-02
	-1	0.27799963	10	-8.92061945E-03
	0	0.84681966	11	-7.73225246E-03
	1	-0.69847577	12	-6.78614593E-03
	2	2.36400139E-02	13	-6.01838599E-03
	3	-8.97463780E-02	14	-5.38521459E-03

M p c on of ope o n e e e

VII.1 Multiplication of matrices in the standard form

e p c on of ce of C de on Zy nd nd p e do d e en ope o n e nd d fo eq e o O N o N ope on n dd on po e o con o e d of e n e nd y e n o ze o e en e n e p od c e o e o d of nd e efo e

$$|| \quad - \quad || \leq \quad (7)$$

e nd de of (7) do n ed y , o e p e f e co p e [4] en e o e one n c n d

VII.2 Multiplication of matrices in the non-standard form

e no o ne n o fo e p c on of e ope o n e non nd d fo ne o e e n e y deco p e e e n e p oce of p c on Le nd e o ope o

$$\quad \mathbf{L} \quad \mathbf{R} \to \mathbf{L} \quad \mathbf{R} \quad (77)$$

en e non nd d fo of nd $\{A_j \; B_j \; \; _j\}_{j \in Z}$ nd $\{A_j \; B_j \; \; _j\}_{j \in Z}$ e co p e e non nd d fo $\{A_j \; B_j \; \; _j\}_{j \in Z}$ of —

e ec e ope o of

n y e e e 7 of o e

(7)

e e

$$\sum_{j}^{n} \left[A_j A_j \quad B_j \quad B_j \; j \quad A_j B_j \quad j \quad A_j \right] \qquad (7)$$

nd

$$n \; n \sum_{j}^{n} P_j B_j P_j \qquad (7)$$

e ope o n e 7 e c n on fo o n e p ce

$$A_j A_j \quad B_j \quad \mathbf{W}_j \to \mathbf{W}_j \qquad (7)$$

$$B_j \; j \quad A_j B_j \quad \mathbf{V}_j \to \mathbf{W}_j \qquad (7)$$

$$j \quad A_j \quad \mathbf{W}_j \to \mathbf{V}_j \qquad (7)$$

nd e ope o n e 7

$$B_j \quad \mathbf{V}_j \to \mathbf{V}_j \qquad (7\;7)$$

e e n 7 d d e f d 7 7 f

of ope on ed e c e e o o e p e e e o n e of
ope on ep p opo on o N
e e n ope o A_j B_j j n tot49 -410.315ac5 0 Td (36su8712.7097 0552

e e o n e e e

e f p c on o of ec on e econd fe o n e of e e
o

VIII.1 An iterative algorithm for computing the generalized inverse

n o de o

p oced e nd e e on n e e e n . e e . n fo . . e co p on e e pe fo ed on n p c o on nd e ed o ne f o L N AC fo co p n e n e deco po on . o e e ed e fo o n f n

$$A_{ij} = \begin{cases} \frac{}{i-j} \\ \end{cases}$$

e e . = N . e cc cy e o d e o $^{-4}$ e en e of X_k e o $^{-4}$ e e y e c y e o ed f e e c e on

Size $N \times N$	SVD	FWT Generalized Inverse	L_2-Error
128×128	20.27 sec.	25.89 sec.	$3\ 1 \cdot 10^{-4}$
256×256	144.43 sec.	77.98 sec.	$3\ 42 \cdot 10^{-4}$
512×512	1,155 sec. (est.)	242.84 sec.	$6\ 0 \cdot 10^{-4}$
1024×1024	9,244 sec. (est.)	657.09 sec.	$7\ 7 \cdot 10^{-4}$
...	...	...	...
$2^{15} \times 2^{15}$	9.6 years (est.)	1 day (est.)	

Le de e e e e o nd c n n e c f nc on c c ope o c n e p e en ed e c en y e fo p e do d . en op e o N e c e nd e e pe fo nce of e o e epo ed p e y

VIII.2 An iterative algorithm for computing the projection operator on the null space.

Le con de e fo o n e on

$$X_{k+} = X_k - X_k$$

$$X = A^* A$$ e

e e A^* e d o n nd c o en

en $-X_k$ con e e o P_{null} c n e o n e e d ec y o y co n n n n n ep e en on fo $P_{null} := - A^* AA^* \dagger A$ e e on o co p e e ene zed n e $AA^* \dagger$ e f p c on o e e e on e f fo de c of ope o e e co p e y e o fo e ene zed n e e po n d ence o e e e doe no eq e co p e y of e n e ope o on y of e po e of e ope o

VIII.3 An iterative algorithm for computing a square root of an operator.

Le de e n e on o con c o A nd A^{-} e e A fo p c y e f d o n nd non ne e de n e ope o e con de e fo o n e on

$$Y_{l+} := Y_l - Y_l X_l Y_l$$

$$X_{l+} := - X_l \quad Y_l A$$

$$Y := - A$$

$$X := -PA$$

7

e e c o en o e e e en e of $- A$ e n $\sqrt{\ }$ e eq ence X_l con e e o A nd Y_l o A^{-} By n $A := V^* D V$ e e D d on nd V n y e y o e fy o X_l nd Y_l c n e en $X_l := V^* P_l V$ nd $Y_l := V^* \ _l V$ e e P_l nd $_l$ e d on nd

$$_{l+} := \ _l - \ _l P_l \ _l$$

P e e

r n f f f d e y A f d f d

VIII.4 Fast algorithms for computing the exponential, sine and cosine of a matrix

e e ponen of o n ope o e ne nd co ne f nc on e on e o e con de ed n ny c c of ope o A n e c e of e ene zed n e e

X Co p n $F(u)$ n e e e e

n ec on e de e f d p e o fo co p n e e n n n e y d en e f nc on nd ep e en ed n e e An po n e p e O n y c e ene ze eo e of M Bony , , , , , , on e p op on of n e of o on of non ne eq on O n e c pp o c o e e no e e e pec de n e of pp c on of o

IX.1 The algorithm for evaluating u^2

e n o o co p e Le p o ec $\in$ **L R** on p ce **V**$_j$; $\in$ **Z** o

$$j = P_j \quad j \in \mathbf{V}_j$$

n o de o deco p e e e e e e op c e e

$$- {}_n = \sum_j^n \Big[P_{j-} - P_j \Big] = \sum_j^n P_{j-} \quad P_j \quad P_{j-} - P_j$$

n $P_{j-} = P_j$ j e o n

$$- {}_n = \sum_j^n P_j \quad j \quad j$$

o

$$= \sum_j^n P_j \quad j \quad \sum_j^n \quad j \quad j \quad n$$

n e e e no n e c on e een d en e nd ; ′ ; ≠ ; ′ o e n e c p po e need fo o e n e n e of e o c e

Befo e p oceed n f e e con de n e p e of e n [redacted]
e e e fo o n e p c e on

$$j \quad k \qquad - \quad -j= \quad j \quad k$$
$$j \quad k \qquad - \quad -j= \quad j \quad k \qquad 7$$
$$j \quad k \quad j \quad k \qquad - \quad -j= \quad j \quad k$$

A o e p od c on e e e e zeo
p nd n e p c y n o [redacted]

$$- \sum_{j}^{n} \sum_{k \in Z} d_k^j \; {}^j_k \qquad \sum_{k \in Z} {}^n_k \; {}^n_k$$

nd n 7 e o n f o e

$$- \sum_{j}^{n} -j= \sum_{k \in Z} d_k^j \; {}^j_k \; {}^j_k \qquad \sum_{j}^{n} -j= \sum_{k \in Z} d_k^j \; {}^j_k \qquad -n= \sum_{k \in Z} {}^n_k \; {}^n_k$$

On deno n

$$d_k^j \quad - \quad -j= + \; d_k^j \; {}^j_k$$
$${}^j_k \quad - \quad -j= \; d_k^j$$
$${}^n_k \quad - \quad -n= \; {}^n_k$$

e e e

$$- \sum_{j}^{n} \sum_{k \in Z} d_k^j \; {}^j_k \qquad \sum_{j}^{n} \sum_{k \in Z} {}^j_k \; {}^j_k \qquad \sum_{k \in Z} {}^n_k \; {}^n_k$$

e no e **if the coe cient d_k^j is zero then there is no need to keep the corresponding average** ${}^j_k$ n o e o d e need o eep e e on y ne e n e e e e e e e coe c en d_k^j o p od c ${}^j_k d_k^j$ e n c n fo en cc cy

of coe c en c need o e o ed y e ed ced f e y o e n fo e p e

$$M^{j;j'}_{WWW} \quad ' \quad - \quad -j'= \int_{-\infty}^{+\infty} \quad j-j' \quad {}^{j-j'}_{k-k'} \quad {}^{j-j'}k-l \quad d$$

o

$$M^{j;j'}_{WWW} \quad ' \quad - \quad -j'= M^{j-j'}_{WWW} \quad - \quad ' \quad j-j' \quad -$$

o e e e o n c n ed c on n e n e of coe c en con e q ence of e f c e coe c en n dec y e d nce r − − ′ e een e e f en e of n c n coe c en d_k^j p opo on o en e of e o N e en e of ope on eq ed o e e e pp n

ce nece y o o e on y o e e $_{k'}^{j}$ c co ne e n c n coe c en d_k^j o p od ce non ze o con on e efo e c en o o e on y o $_{k'}^{j}$ fo c e e e e coe c en d_k^j c $| - '| \leq$ nd e p od c $_{k'}^{j} d_k^j$ o e e e od of cc cy e n e need o o e e e on y n e ne o ood of n e

e n e of ope on fo e p nd n of e econd e n n o e ee p opo on o e n e of n c n en e nd e e e co p e e y o fo

Remark. e o fo e on n e e e o o e e e p od c of o f nc on nce $\frac{}{4}$

IX.2 The algorithm for evaluating F(u)

Le e n n n e y d en e f nc on n o de o deco p e e e e n e e cop c e

$$- \quad {}_{n} \quad \sum_{j}^{n} \quad P_{j} - \quad - \quad P_{j} \qquad (7)$$

p nd n e f nc on n e y o e e e po n y e P d 7 7 e de p

e no e e e no e e econd de e of n e e e no e en
de e n n nd con de n e e nde of e e e n
n e o e e o n e e of M Bony e e o e o e e e
y oo e n n Bony e nce e e nde f c o j n e d
of j e y eep o e e o e e e nde y oo
no c e po n f e e n d n e n co p n
o epe ed pp c on of e o fo c en o co p e o
f nc on o e e e e e e n y c d n e n con de n n
p c y n eo e c c e z n e

efe ence

. , B A pe p e ep e en on of oo ne ope o D e Y e n e y

. , C e L een d nd o n A f d p e po e o fo p ce on **SIAM Journal of Scienti c and Statistical Computing** e Y e n e y ec n c epo YALeB DO o e

. e A Co en D ec e nd C .

. , M e e of f eq ency c nne deco po on of e nd e e ode ec n c epo e Co n n e of M e c c ence Ne Yo n e y

. , Y Meye Le c c en q e e onde e e e e o en q d e C MAD n e e D p ne

. , Y Meye nc pe d nce de e e enne e e e d ope e nw6ti3T