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# Compactly Supported Wavelets Based on Almost Interpolating and Nearly Linear Phase Filters (Coiflets)

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New compactly supported wavelets for which both the scaling and wavelet functions have a high number of vanishing moments are presented. Such wavelets are a generalization of the so-called coiflets and they are useful in applications where interpolation and linear phase are of importance. The new approach is to parameterize coiflets by the first moment of the scaling function. By allowing noninteger values for this parameter, the interpolation and linear phase properties of coiflets are optimized. Besides giving a new definition for coiflets, a new system for the filter coefficients is introduced. This system has a minimal set of defining equations and can be solved with algebraic or numerical methods. Examples are given of the various types of coiflets that can be obtained from such systems. The corresponding filter coefficients are listed and their properties are illustrated. © 1999 Academic Press

## 1. INTRODUCTION

Among compactly supported wavelets for $L^2(\mathbf{R})$ a family known as *coiflets* has a number of properties that make it particularly useful in numerical analysis and signal processing [1, 8, 9]. Coiflets allow for both the scaling and the wavelet functions to have a high number of vanishing moments and, as we show here, the associated low-pass filters are almost interpolating and nearly linear phase within the passband. In 1989, R. Coifman

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suggested the design of orthonormal wavelet systems with vanishing moments for both the scaling and the wavelet functions. They were first constructed by Daubechies [9] and she named them *coiflets*.

In [1] shifted vanishing moments for the scaling function $\varphi$ were used to obtain *one point quadratures*

$$f(x) \approx \sum_{k \in \mathbf{Z}} f(x_k)\, \varphi(x - k); \tag{1.1}$$

where $f$ is a sufficiently smooth function on the multiresolution space $V_0$ and $\{f(x_k)\}$ are good approximations of the coefficients of $f$ in the expansion.

Since in [1] both matrices and operators were considered, the points $\{x_k\}$ were chosen to be $x_k = k +$ , where is an integer. This “shift” corresponds to the first moment of the scaling function $\varphi$,

$$= \int_{\mathbf{R}} x\, \varphi(x)\, dx. \tag{1.2}$$

Note that is not the center of mass because $\varphi(x)$

wavelet . The key to our approach is to insist on a reasonable approximation to linear phase only in the passband of the associated low-pass filter $m_0$.

It is well known that the properties defining coiflets can be easily described in terms of the coefficients $\{h_k\}$ of $m_0$. The conditions on $\{h_k\}$ turn out to be dependent [14], and one of the goals of this article is to derive a system that is free of redundant equations. To obtain such a system, we perform a change of variables on $\{h_k\}$ via a linear transformation that has the shift as a parameter. This defining system is partly linear and partly quadratic. For filter lengths up to 20 the system can be explicitly solved via algebraic methods like

We also refer to such $H$ as a QMF. As a consequence of (2.3), $H$ satisfies the following functional equation:

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1})$$

In terms of the symbol $H$, (3.1) requires that

$$\sum_j (-1)^j j^k h_j = 0 \qquad \text{for } 0 \le k < M; \tag{3.3}$$

or equivalently, the factorization

$$H(z) = \left(\frac{1+z}{2}\right)^M Q(z); \tag{3.4}$$

where $Q(-1) \neq 0$.

As pointed out in the Introduction, we are interested in vanishing (shifted) moments of the scaling function

$$\mathcal{M}_{\varphi;k} =$$

Equations (3.6) and (3.8) imply that the following four conditions, valid for all $k$; $0 \le k < N$, are equivalent:

$$\mathcal{M}_k^{\varphi} = \int_{\mathbf{R}} x^k \varphi(x)\,dx = \alpha^k; \tag{3.12}$$

$$\mathcal{M}^{\varphi}_{\alpha;k} = \int_{\mathbf{R}} (x-\alpha)^k \varphi(x)\,dx = \delta_{k0}; \tag{3.13}$$

$$\mathcal{M}^{h}_{\alpha;k} = \sum_j (j-\alpha)^k h_j = \delta_{k0}; \tag{3.14}$$

$$\mathcal{M}_k^{h} = \sum_j j^k h_j = \alpha^k. \tag{3.15}$$

Therefore, imposing moment conditions for either the wavelet or the scaling function amounts to finding a QMF $H$ with moment conditions for its sequence of coefficients. In particular, the first moment of $\varphi$, as defined in (1.2), equals the derivative of $H$ at one,

$$\alpha = H'(1).$$

On the other hand, (2.1) forces $|H(z)| \le 1$ for $z$ on the unit circle. These last two properties allow us to show that the value $\alpha$ should be within the support of $\varphi$. Observe that this result is not evident since $\varphi$ is not a positive function.

PROPOSITION 3.2. *Let* $H(z) =$

$$\sum_{j=0}^{L-1} (-1)^j j^k h_j =$$

*Proof.* Applying the operator $.xD/^n$ (defined in the Appendix) at $z$ D 1 to the QMF equation (2.5), or taking derivatives at D 0 in (2.2

its values on a shifted dyadic grid:

$$p(x) = \sum_k p\left(\frac{+k}{2^n}\right)\varphi(2^n x - k).$$

Since at some scale any smooth function can be well approximated by polynomials, we have the almost interpolating property discussed in the Introduction.

Here we see the advantage of having both *M* *a neighborhood of* $= 0$

$$H(e^{i}) = a_H()e^{ip_H()} \quad \textit{and} \quad 0() = a_{0}$$

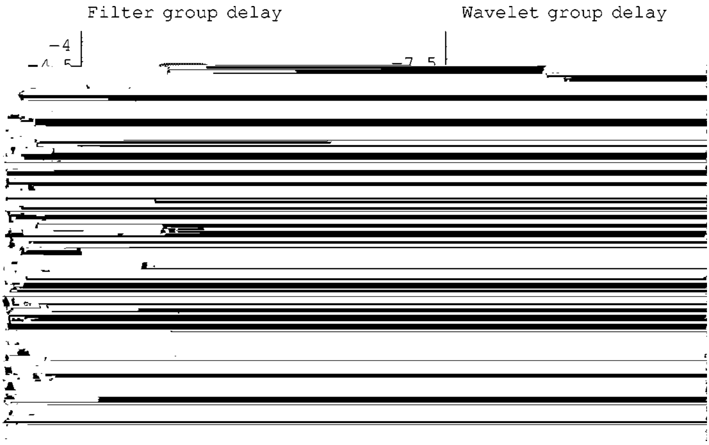


**FIG. 2.** Comparison between the group delays of $m_0$ and 0. Maximal coiflet for wavelet: length 18, case b (top) and Daubechies' least asymmetric filter of the same length (bottom).

least asymmetric filter in Fig. 2 is defined as the maximally flat filter whose phase is as linear as possible within the whole band T ; U. See [8 we obtain the *polyphase* equation

$$H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = \frac{1}{2}. \tag{7.6}$$

The problem of finding a solution $H$ of the QMF equation (2.5) is thus replaced by finding the solutions $H_0$ and $H_1$ of the *polyphase* equation. Instead of performing two operations on the variable $z$ in (2.5), namely $-z$ and $z^{-1}$, in (7.6) we only have $z^{-1}$.

## 8. THE CONSTRUCTION OF COIFLETS

Recall that we can write any polynomial QMF as $H(z) = \sum_{k=0}^{L-1} h_k z^k$, where $h_0 h_{L-1} \neq 0$.

We describe a system for coiflets not in terms of $\{h_k\}$ but in terms of the new variables

$$a_k = \frac{1}{k!}\sum_j \left(j - \frac{\alpha}{2}\right)^k h_{2j} \quad \text{and} \quad b_k = \frac{1}{k!}\sum_j \left(j - \frac{\alpha - 1}{2}\right)^k h_{2j+1},$$

where $0 \leq k \leq l$, and $l = \frac{1}{2}(L-2)$. The transformation from $\{h_k\}$ to $\{a_k, b_k\}$ is linear and parameterized by $\alpha$. As before, $\alpha = \sum_j j h_j$ is the first moment of $\varphi$.

For what follows, it is more convenient to describe $a_k$ and $b_k$ for arbitrary $k \geq 0$, using the operator $xD$. W 2 k r i k r i k r 1 0 n T r b i t i s ( 0 , ) - t ( g 2 3 w 9 j / F 1 0 1 T

$$= \frac{1}{n!}(xD)^n\left(x^{-1/2}H_0(x)x^{1/2}H_0(x^{-1}) + x^{\cdot(1/2)}H_1(x)x^{\cdot(-1/2)}H_1(x^{-1})\right)(1)$$

$$= \sum_{k=0}^{n} (-1)^k (a_{n-k}a_k + b_{n-k}b_k) \tag{8.1}$$

for $0 \le n \le L-2$.

If $n$ is odd, the previous equation is always satisfied and then, as we remarked earlier, $\frac{L}{2}$ equations are enough to characterize a QMF of length $L$.

*8.2. Linear Conditions*

We now discuss how to rewrite the (linear) conditions (4.1) and (4.2) for coiflets in terms

### 8.3. *A System for Coiflets*

Combining (8.1) and (8.4), the system for coiflets can be written in terms of the unknowns $\{\alpha; a_m; \dots; a_l; b_m; \dots; b_l\}$,

$$\frac{1}{2}\delta_{n0} = a_n + b_n + \sum_{k=m}^{n-1} (-1)^k a_{n-k}$$

*8.3.3. Coiflets with integer shifts.* Coiflets for *integer* choices of the shift were first computed by Daubechies [9]. In all cases that we computed, coiflets with integer shifts were always nonmaximal. In Table 1 we list, for different lengths $L$, the range of possible integer shifts in .0;9 09n

An equivalent system, obtained via Gröbner bases where is treated as parameter is function (by setting $a_3$ D $b_3$). In the latter case, (9.3) becomes

**TABLE 2**
**Coiflet Filters of Length 8**

| | $k$ | $h_k$ | | $k$ | $h_k$ |
|---|---|---|---|---|---|
| $M$ D 3 | 0 | 0:00899863735774892 | $M$ D 3 | 0 | 0:03952785122359428 |
| $N$ D 5 | 1 | 0:02054552466216258 | $N$ D 5 | 1 | 0:1271031281675352 |
| D $_1$ | 2 | 0:2202099211463259 | D $_2$ | 2 | 0:5323389066059403 |
| Case Na | 3 | 0:5701914465849665 | Case Nb | 3 | 0:440002251136967 |
| MAXIMAL | 4 | 0:3422577968313942 | MAXIMAL | 4 | 0:005981694132267174 |
| | 5 | 0:07306459213264614 | | 5 | 0:07120132136770919 |
| | 6 | 0:05346908061997128 | | 6 | 0:01317063874992116 |
| | 7 | 0:02341867020984207 | | 7 | 0:004095942063206933 |
| $M$ D 3 | 0 | 0:1646660519380485 | $M$ D 3 | 0 | 0:3040839480619514 |
| $N$ D 3 | 1 | 0:5074101320413008 | $N$ D 3 | 1 | 0:414464867958699 |
| D 1 | 2 | 0:4435018441858542 | D 1 | 2 | 0:02524815581414562 |
| Case 1a | 3 | 0:02223039612390291 | Case 1b | 3 | 0:2566053961239029 |
| | 4 | 0:1310018441858543 | BAD | 4 | 0:2872518441858542 |
| | 5 | 0:02223039612390291 | | 5 | 0:2566053961239029 |
| | 6 | 0:02283394806195145 | | 6 | 0:1165839480619514 |
| | 7 | 0:007410132041300974 | | 7 | 0:085535132041301 |
| $M$ D 3 | 0 | 0:01938529090153145 | $M$ D 3 | 0 | 0:0850102909015314 |
| $N$ D 3 | 1 | 0:1854738954507657 | $N$ D 3 | 1 | 0:1332761045492342 |
| D 2 | 2 | 0:5581558727045942 | D 2 | 2 | 0:2449691272954056 |
| Case 2a | 3 | 0:3810783136477028 | Case 2b | 3 | 0:5376716863522972 |
| | 4 | 0:05815587270459436 | UGLY | 4 | 0:2550308727045943 |
| | 5 | 0:06857831364770281 | | 5 | 0:2251716863522971 |
| | 6 | 0:01938529090153145 | | 6 | 0:0850102909015314 |
| | 7 | 0:002026104549234272 | | 7 | 0:05422389545076572 |
| $M$ D 3 | 0 | 0:05191993211769211 | $M$ D 3 | 0 | 0:01058006788230788 |
| $N$ D 3 | 1 | 0:0234375 | $N$ D 3 | 1 | 0:0234375 |
| D 3 | 2 | 0:3432597963530763 | D 3 | 2 | 0:2192402036469236 |
| Case 3a | 3 | 0:5703125 | Case 3b | 3 | 0:5703125 |
| | 4 | 0:2192402036469236 | | 4 | 0:3432597963530763 |
| | 5 | 0:0703125 | | 5 | 0:0703125 |
| | 6 | 0:01058006788230788 | | 6 | 0:05191993211769211 |
| | 7 | 0:0234375 | | 7 | 0:0234375 |

*Note.* $_1$ D 2:977273091796802, $_2$ D 2:239549738364678.

## *9.2. Coiflets of Length 18*

A similar analysis can be done for filters of length 18. In Table 4, we present a summary of our findings by listing the filter coefficients for two cases: coiflets with integers shifts and maximal coiflets. Filter coefficients are listed in Table 6.

Even at higher numbers of vanishing moments and different lengths, we still found UGLY and BAD filters. They always correspond to coiflets with integer shifts, but it is not a peculiarity of that case. Varying , we found regions of nonmaximal coiflets with a similar behavior.

**TABLE 3**
**Summary of All Maximal Coiflets and Coiflets with Integer Shifts for Length 8**

| Filter | | $M$ | $N$ | | $\|Q\|$ | $2^{M-1}$ | Remarks |
|---|---|---|---|---|---|---|---|
| Na | 2.97727 | 3 | 5 | 1.45584 | 2.8764 | 4 | |
| Nb | 2.23955 | 3 | 5 | 1.44599 | 2.94511 | 4 | |
| Ma | 1.00539 | 4 | 3 | 1.77557 | 5.91608 | 8 | Daubechies' Extremal Phase |
| Mb | 2.98547 | 4 | 3 | 1.77557 | 5.91608 | 8 | Daubechies' Least Asymmetric |
| 1a | 1 | 3 | 3 | 1.77528 | 2.16403 | 4 | |
| 1b | 1 | 3 | 3 | 0.14666 | 14.9356 | 4 | BAD |
| 2a | 2 | 3 | 3 | 1.42232 | 3.11099 | 4 | |
| 2b | 2 | 3 | 3 | 0.93596 | 6.91099 | 4 | UGLY |
| 3a | 3 | 3 | 3 | 1.77341 | 2.16473 | 4 | |
| 3b | 3 | 3 | 3 | 1.46353 | 2.82288 | 4 | |

*Note*. Coefficients are listed in Table 2. The maximal case for wavelets coincides with Daubechies' maximally flat filters.

In Figs. 11 and 12, we plotted $|m_0|$ and ′ for the cases 6c (UGLY) and 5b (BAD) with length 18. The cases 7d and 6d, as listed in Table 4, exhibit a similar behavior. Even though their filter moduli do not oscillate as much as their counterparts of length 8, their behavior is clearly different than those for which $\|Q\|$ remains below $2^{M-1}$. As an example of the latter situation, consider the filter 7c. The associated wavelet has only six vanishing moments, but its Sobolev exponent is higher than the exponent for Daubechies' wavelets which have nine vanishing moments.

Note that in all the plots for wavelets in the Fourier domain, the support of the functions is actually wider than shown.

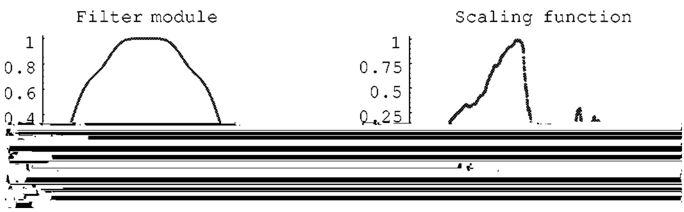


**FIG. 3.** Integer shift coiflet: length 8, shift 2, case b (UGLY). Plots of absolute value of filter $m_0$ and scaling function.

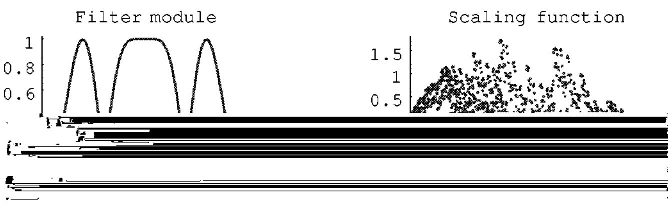


**FIG. 4.** Integer shift coiflet: length 8, shift 1, case b (BAD). Plots of absolute value of filter $m_0$ and scaling function.

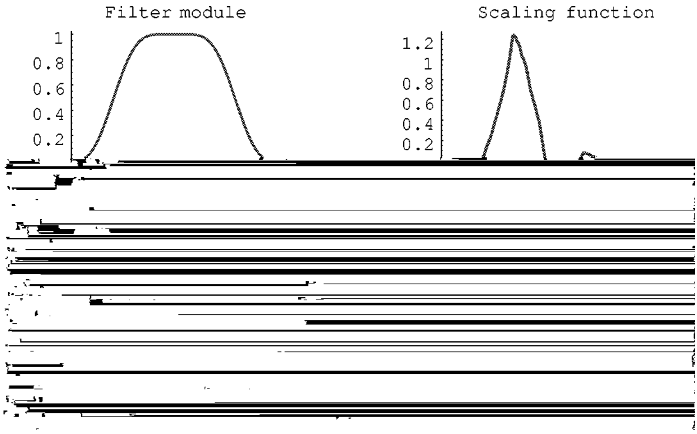


**FIG. 5.** Integer shift coiflet: length 8, shift 2, case a. Plots of scaling function and filter $m_0$.

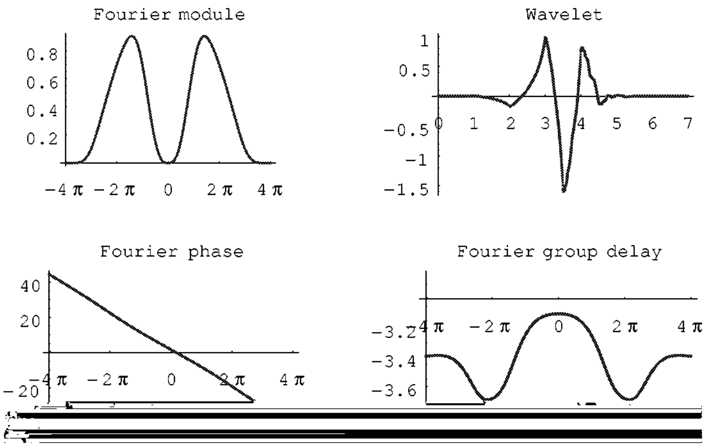


**FIG. 6.** Integer shift coiflet: length 8, shift 2, case a. Plots of wavelet function in both time and Fourier domain (absolute value, phase, and group delay).

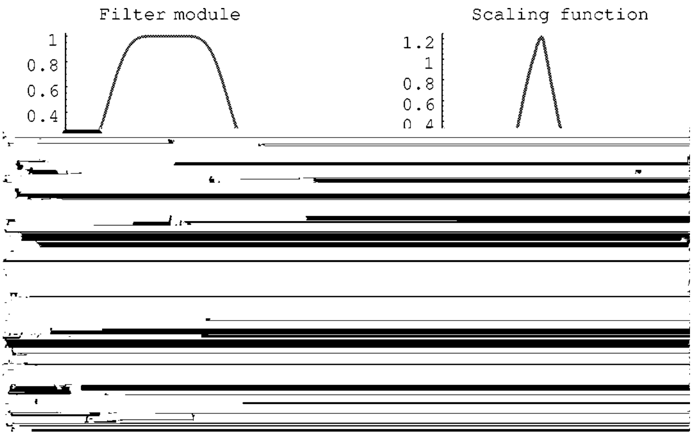


**FIG. 7.** Integer shift coiflet: length 8, shift 3, case a. Plots of scaling function and filter $m_0$.

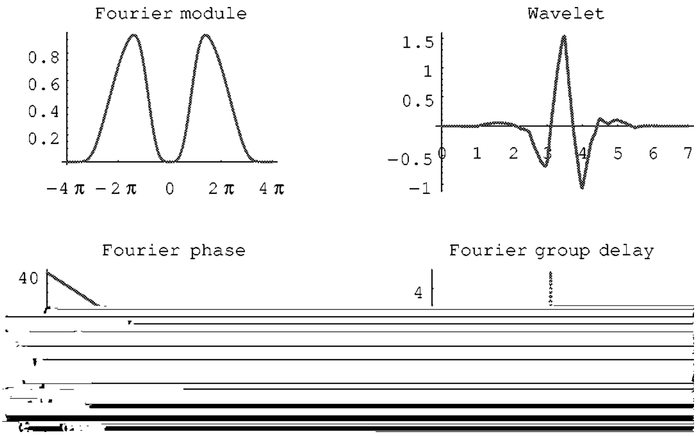


**FIG. 8.** Integer shift coiflet: length 8, shift 3, case a. Plots of wavelet function in both time and Fourier domain (absolute value, phase, and group delay).

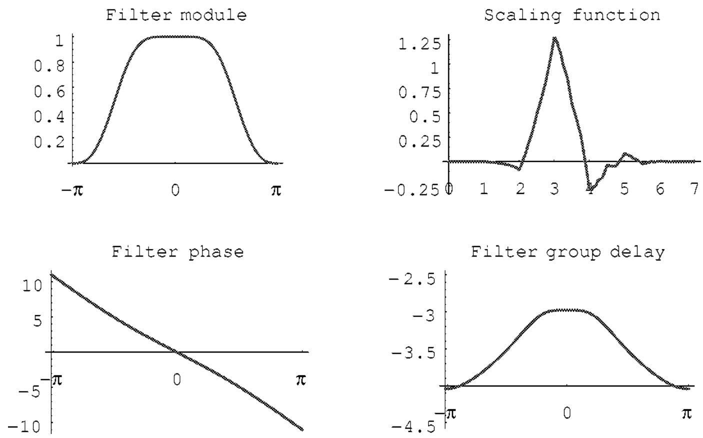


**FIG. 9.** Maximal coiflet for scaling function: length 8, shift 2:9773. Plots of the scaling function and filter $m_0$.

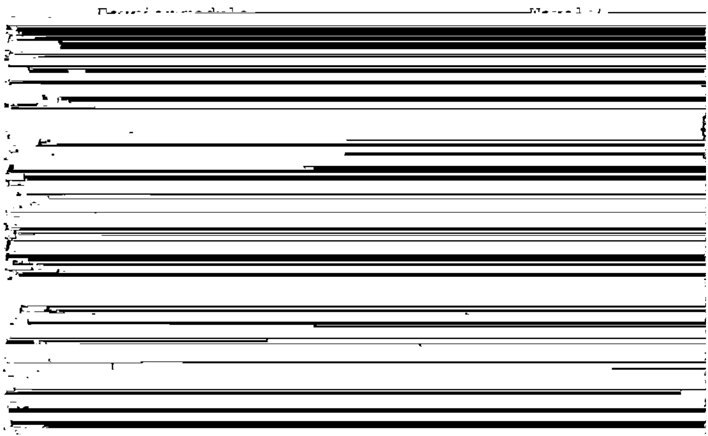

**FIG. 10.** Maximal coiflet for scaling function: length 8, shift 2:9773. Plots of the wavelet function in both time and Fourier domain (absolute value, phase, and group delay).

**TABLE 4**
**Summary of All Maximal Coiflets, Coiflets with Integer Shifts, and Two Daubechies' Maximally Flat Filters for Length 18**

| Filter | | $M$ | $N$ | | k$Q$k | $2^{M}$ 1 | Remarks |
|---|---|---|---|---|---|---|---|
| Na | 7.81041 | 6 | 9 | 2.5149 | 16.5942 | 32 | Listed in Table 5 |
| Nb | 7.1771 | 6 | 9 | 2.49853 | 17.2438 | 32 | Listed in Table 5 |
| Ma | 5.94301 | 7 | 7 | 2.74543 | 33.9874 | 64 | Listed in Table 5 |
| Mb | 4.5681 | 7 | 7 | 2.71944 | 36.2534 | 64 | Listed in Table 5 |

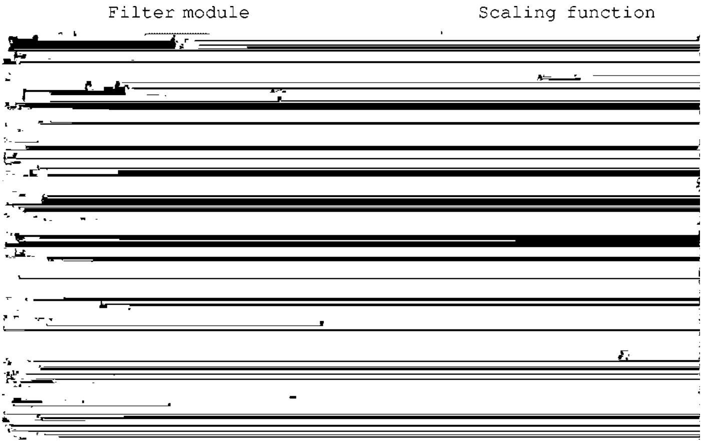


**FIG. 13.** Maximal coiflet for scaling function: length 18, shift 7.1771. Plots of filter $m_0$ and scaling function.

## 10. CONCLUSION

The approach taken in this paper allows one to construct and classify coiflets, which are wavelets with a high number of vanishing moments for both the scaling and wavelet functions. Coiflet filters are useful in applications where interpolation and linear phase are of particular importance.

We introduced a new system for FIR coiflets. In all cases investigated, the system had a minimal set of defining equations. For filters of length up to 20, the system can be solved explicitly, and the filter coefficients can thus be accurately determined. For longer filters we applied numerical methods to compute some solutions. For a few specific examples we studied the properties of coiflets corresponding to both integer and noninteger values of the first moment of the scaling function. Nevertheless, the problem of the existence of coiflet filters of arbitrary length and their full classification remains open.

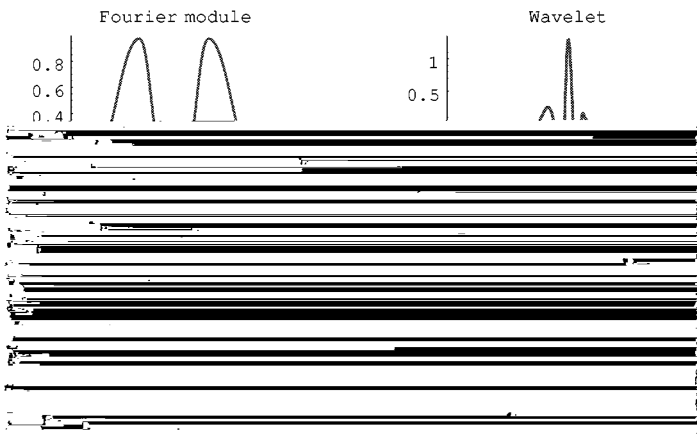


**FIG. 14.** Maximal coiflet for scaling function: length 18, shift 7.1771. Plots of wavelet function in both time and Fourier domain (absolute value, phase, and group delay).

**TABLE 5**
**Coiflet Filters of Length 18: Maximal Case**

| | $k$ | $h_k$ | | $k$ | $h_k$ |
|---|---|---|---|---|---|
| $M$ D 6 | 0 | 0:00006423105557385401 | $M$ D 6 | 0 | 0:0002036914946771235 |
| $N$ D 9 0: | 1 | | | | |

From (A.1) and (A.3)

$$z^n D^n f(z) = \sum_{k=0}^{n} s_k^n (xD)^k f(z). \tag{A.4}$$

Note that for a polynomial of degree $r$, it is not true that $(xD)^n P(1)$ is zero for $n > r$. However, these values are linear combinations of $(xD)^n P(1)$ for $n \le r$, as we show in the

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